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ICE MECHANICS

by

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PREFACE

During the final period November 1981 - September 1983, the contract continued to support Dr. U. Spring as a senior research associate, followed by a gap of 3 months, then supported Dr. H. T. Williams as a senior research associate.

This report is presented in two parts, I and II. Part I has now been published in Cold Regions Science and Technology, 6 (1983) 185-193, being a continuation of my work with Dr. Spring on integral representations for the viscoelastic deformation of ice. It is concerned with different structures of integral representation, and how the respective kernels are correlated with conventional constant stress and constant strain-rate data.

Part II is a paper written by Dr. Williams, which will be published by Cold Regions Science and Technology during 1984. Dr. Williams developed an alternative integral representation which reflected the non-monotonic nature of the response to constant stress without requiring an explicit non-monotonic kernel function of time. The latter is shown to have unsatisfactory physical implications as well as previously noted application difficulties. The new representation is also shown to be consistent with a variety of different response features.

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Report is in two parts, as follows:

Integral Representation for the Viscoelastic Deformation of Ice. Various single integral representations which describe non-linear viscoelastic response are examined with regard to the types of test required to determine the respective kernels. A strain formulation determined by constant uni-axial stress response typical of ice, and its predictions for constant strain-rate response, are reviewed, showing that the latter are sensitive to kernel

detail. An alternative stress formulation which is determined by constant strain-rate response is constructed, and it is shown that the predicted strain and strain-rate responses at constant stress are compatible with the typical responses exhibited by ice. Single Integral Representations in Ice Mechanics. A single integral viscoelastic constitutive equation for ice is developed which possesses significant theoretical and practical advantages over previously suggested equations of this type (Spring and Morland, 1983). The theory is specialized to the case of small strain uniaxial compression and the resulting constitutive equation is shown to verify the relations between experimental data obtained in constant load (CL) creep tests and constant displacement rate (CD) "strength" tests conjectured in Mellor and Cole (1982) and demonstrated in Mellor and Cole (1983).

PART I

Integral Representations for the Viscoelastic

Deformation of Ice

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Abstract

Various single integral representations which describe
non-linear viscoelastic response are examined with regard to
the types of test required to determine the respective kernels.
A strain formulation determined by constant uni-axial stress
response typical of ice, and its predictions for constant
strain-rate response, are reviewed, showing that the latter
are sensitive to kernel detail. An alternative stress
formulation which is determined by constant strain-rate response
is constructed, and it is shown that the predicted strain and
strain-rate responses at constant stress are compatible with
the typical responses exhibited by ice.

1.   Introduction

The viscoelastic properties of ice are commonly exhibited by the creep response at constant stress, and the stress response at constant strain-rate, in uni-axial compressive stress tests at constant temperature, reviewed by Mellor (1980). While detailed results over the large temperature range of practical concern (230K → 273K) have not been determined, the main qualitative features of the responses up to strains of a few percent are known.   Typical responses are shown in Fig. 1 where σ is the uni-axial compressive stress, e is the axial engineering compressive strain defined as the decrease in length per unit initial length, t denotes time, and $\dot{e}$ is the strain-rate. At constant stress σ , Fig. 1a, there is an initial elastic strain jump $e_e(\sigma) = \sigma/E_o$, where E_o is the Young's modulus at zero stress, of order $10^{10} Nm^{-2}$, followed by a primary decelerating creep $(\ddot{e} < 0)$ until time $t_m(\sigma)$ when $\ddot{e} = 0$, $\dot{e} = r_m(\sigma)$, $e = e_m(\sigma)$, then a tertiary accelerating creep $(\ddot{e} > 0)$. The minimum strain-rate $r_m(\sigma)$, also referred to as secondary or steady state creep, is significantly non-linear in σ and highly temperature dependent, but the strain $e_m(\sigma)$ is approximately 0·01 over a moderate stress range $0 \to 10^6 Nm^{-2}$. By comparison, the initial strain jump $e_e << e_m$ over this stress range, and so is negligible in applications involving tertiary creep. The associated strain-rate shown in Fig. 1b illustrates more clearly the non-monotonic response features not common in other rheological models.   At constant strain-rate $\dot{e} = r$, Fig. 1c, the stress increases continuously from zero

to a maximum $\sigma_M(r)$ at time $t_M(r)$ and strain $e_M(r) = rt_M(r)$, then decreases. The strain $e_M(r)$ is also found to be approximately 0·01 over a moderate range of r.

These multi-type test responses have been modelled by viscoelastic differential operator relations of fluid type (Morland and Spring 1981) involving stress, stress-rate, strain-rate, and strain-acceleration terms, and of solid type (Spring and Morland 1982) involving stress, stress-rate, strain, and strain-rate terms. In the fluid relation the constant stress and constant strain-rate responses are neither completely independent nor completely dependent, so reflect some common and some distinct properties. In the solid relation these two responses are completely independent, and, in fact, are insufficient to determine the model without further reductions. Mellor (1980) conjectured that the two types of response should be dependent, which is the familiar linear result. While the differential fluid relation is closer to this conjecture, it cannot model anisotropy or strain jumps. In both types it is necessary to incorporate dependence of one or more response coefficients on both stress and strain-rate or strain respectively, and covering adequate stress-strain-rate or stress-strain domains by practical test programmes is non-trivial in uni-axial stress, and the difficulty is compounded in the multi-axial tests required to describe a three-dimensional response.

In contrast, an integral representation expresses the stress (stress formulation) explicitly in terms of strain history or the strain (strain formulation) explicitly in terms of

stress   - 4 -

history. The material properties are therefore defined by
integral kernel functions of strain history or of stress
history, but not both together. With the above differential
relations (Morland and Spring 1981, Spring and Morland 1982),
the strain-rate (strain) response to an applied stress
history, and the stress response to an applied strain-rate
(strain) history, are determined by solutions to differential
equations in time, while with integral relations they are
determined by direct integration or by solutions to Volterra
integral equations. Integral representations may therefore
offer numerical advantages in the solution of boundary-value
problems, by not increasing the order of stress derivatives,
as well as a more direct correlation with test data. An
investigation of different single integral representations to
describe given constant uni-axial stress response has been
presented by Morland and Spring (1982). Such strain formulations
define the uni-axial response completely, and in particular,
the stress histories required to maintain constant strain-rate
are determined by Volterra integral equations. Numerical
solutions based on the shape of respone illustrated in
Figs la and lb show that the constant strain-rate response
is acceptable only for one of the representations, and is then
sensitive to kernel detail, not always giving the required
response shape illustrated in Fig. 1c. One conclusion is that
the single integral strain representation should be correlated
approximately with both constant stress and constant strain-rate
responses by some optimisation procedure, and not correlated
exactly with the constant stress response, but such a scheme
has yet to be devised.

The above results suggest that the constant strain-rate
response of ice may provide a better basis for a single
integral representation. Since the kernels of the stress
and strain formulations discussed by Morland and Spring (1982),
depending on strain history and stress history respectively,
cannot be determined by constant strain-rate response, we
now investigate two alternative stress formulations with
kernels depending on strain-rate history. One defines a
viscoelastic solid which exhibits no strain jump when a stress
jump is applied, which is a good approximation in many
situations, and the second defines a viscoelastic fluid. A
stress formulation is also the most convenient for substitution
in momentum or equilibrium equations. Both respresentations
are equivalent in the small strain, small rotation,
approximation. Kernels describing the shape of uni-axial
stress response illustrated in Fig. 1c are constructed,
and the associated creep and strain-rate responses at
constant stress are determined numerically from the resulting
Volterra integral equation. It is found that these responses
are consistent with the shapes illustrated in Figs 1a and 1b,
and we conclude that this form of single integral
representation is a better description of the uni-axial
multi-type responses illustrated in Fig. 1 than the previous
strain formulation determined by constant stress response.

2. Integral representations

A general frame indifferent representation of a

viscoelastic material is

$$\hat{\sigma}(t) \equiv R^T(t)\sigma(t)R(t) = \mathcal{F}[E(\overset{t}{\underset{-\infty}{\tau}})] , \tag{1}$$

where σ is the Cauchy stress, E is a strain defined by $\frac{1}{2}(F^T F - 1)$, and F and R are the deformation gradient and rotation from a given reference configuration {see for example, Dill (1975)}. That is, the rotated stress $\hat{\sigma}$ at time t depends on the history of strain $E(\tau)$, $-\infty < \tau \le t$, which we designate a stress formulation. It is supposed that the relation (1) can be inverted for viscoelastic materials (thought not for viscous fluids) to give a strain formulation

$$E(t) = \mathcal{H}[\hat{\sigma}(\overset{t}{\underset{-\infty}{\tau}})] \tag{2}$$

in which the strain at time t depends on the history of the rotated stress. Any material symmetries of the reference configuration impose restrictions on the tensor functionals $\mathcal{F}$ and $\mathcal{H}$. If the material is incompressible, then the strain satisfies the constraint

$$det(1 + 2E) = 1 , \tag{3}$$

and the stress is determined only within an arbitrary additive isotropic pressure; that is, $\hat{\sigma}$ is replaced by $\hat{\sigma} + p1$ in the relations (1) and (2) where p is an arbitrary scalar field.

Multiple integral expansions of the stress functional $\mathcal{F}$ with kernels depending only on the time lapse $t - \tau$

were first proposed by Green and Rivlin (1957), and have received considerable attention. Even with the commonly adopted third order truncation and restriction to isotropic material, the necessary test programme to determine the kernels is prohibitive. A detailed account is given by Lockett (1972). While a truncated expansion is an exact frame indifferent theory, and may describe highly non-linear response of a real material, its motivation is that the current strain and weighted integration of past strain (fading memory) are small. Furthermore, the use of multiple integral expansions in the solution of boundary-value problems is not too appealing, particularly if a high order truncation is required. An alternative proposal by Pipkin and Rogers (1968) is that the first single integral term should be a good approximation, and reflect any significant non-linearity by including general dependence of the kernel on the past strain in a stress formulation (1), and on the past stress in a strain formulation (2). Such single integral representations are

$$\hat{\underset{\sim}{\sigma}}(t) = \underset{\sim}{G}[\underset{\sim}{E}(t),0] + \int_{-\infty}^{t} \dot{\underset{\sim}{G}}[\underset{\sim}{E}(\tau),t-\tau]d\tau \ , \tag{4}$$

$$\underset{\sim}{E}(t) = \underset{\sim}{J}[\hat{\underset{\sim}{\sigma}}(t),0] + \int_{-\infty}^{t} \dot{\underset{\sim}{J}}[\hat{\underset{\sim}{\sigma}}(\tau),t-\tau]d\tau \ , \tag{5}$$

where the relaxation function $\underset{\sim}{G}[E,t]$ and creep function $\underset{\sim}{J}[\hat{\sigma},t]$ are tensor functions of strain $\underset{\sim}{E}$ and stress $\hat{\underset{\sim}{\sigma}}$ respectively, and of a positive scalar time argument. $\dot{\underset{\sim}{G}}, \dot{\underset{\sim}{J}}$ denote derivatives of $\underset{\sim}{G}$ and $\underset{\sim}{J}$ respectively with respect to the second (scalar) argument.

It is supposed that the stress after maintained zero strain is zero, and the strain after maintained zero stress is zero, so

$$\underset{\sim}{G}(\underset{\sim}{0},t) \equiv 0 , \quad \underset{\sim}{J}(\underset{\sim}{0},t) \equiv 0 . \tag{6}$$

The stress given by the representation (4) for a constant strain history $\underset{\sim}{E}(\tau) = \underset{\sim}{E}_O H(\tau)$, where $H(t)$ is the Heaviside step function, is

$$\underset{\sim}{E}(\tau) = \underset{\sim}{E}_O H(\tau) : \quad \underset{\sim}{\hat{\sigma}}(t) = \underset{\sim}{G}[\underset{\sim}{E}_O,t] , \tag{7}$$

and the strain given by the representation (5) for a constant (rotated) stress history $\underset{\sim}{\hat{\sigma}}(\tau) = \underset{\sim}{\hat{\sigma}}_O H(\tau)$ is

$$\underset{\sim}{\hat{\sigma}}(\tau) = \underset{\sim}{\hat{\sigma}}_O H(\tau) : \quad \underset{\sim}{E}(t) = \underset{\sim}{J}[\underset{\sim}{\hat{\sigma}}_O,t] . \tag{8}$$

Thus, a programme of constant strain tests determines the kernel $\underset{\sim}{G}(\underset{\sim}{E}_O,t)$ over the range of $\underset{\sim}{E}_O$ covered, and hence determines the single integral representation (4) completely. The stress $\underset{\sim}{\hat{\sigma}}(t)$ then predicted for arbitrary strain history $\underset{\sim}{E}(\tau)$, $-\infty < \tau \leq t$, is given by direct integration, and the strain $\underset{\sim}{E}(t)$ predicted for arbitrary stress history $\underset{\sim}{\hat{\sigma}}(\tau)$, $-\infty < \tau \leq t$, is the solution of a non-linear Volterra integral equation. Similarly, a programme of constant stress tests determine the kernel $\underset{\sim}{J}(\underset{\sim}{\hat{\sigma}}_O,t)$ over the range of $\underset{\sim}{\hat{\sigma}}_O$ covered, and hence determines the single integral representation (5) completely. Pipkin and Rogers (1968) show how a sequence of multiple integral corrections can be constructed if the responses to multi-step strain or

stress tests do not correspond exactly to the predictions of
the lower order representations, starting with the relations
(4) and (5) respectively, but this is again a prohibitive
programme (Lockett 1972). Clearly single integral representations
are the most tractable for stress and deformation analysis, and
Pipkin and Rogers (1968) have demonstrated for some materials
that the strain formulation (5) in uni-axial stress determined
by a single-step constant stress test predicts good agreement
with the responses to multi-step loads. If, however,
multi-type test data are more common, such as the constant
stress and constant strain-rate responses of ice illustrated
in Fig. 1, there is no direct scheme analogous to the
multi-step test case to correct a single integral approximation.
Morland and Spring (1982) have investigated the strain
formulation (5) in uni-axial stress with varying scalar
kernels $J[\sigma, t]$ compatible with the shape of the small strain
creep responses Figs 1a and 1b. Numerical solutions of the
non-linear Volterra integral equations for the corresponding
stress histories $\sigma(t)$ required to maintain constant
strain-rate in $t \geq 0$ show that the required shape of
response Fig. 1c is not always obtained. That is, the response
to constant strain-rate predicted by a single integral
representation (5) is sensitive to kernel variation, which
implies that a kernel $J[\sigma, t]$ determined solely by constant
stress data may yield unsatisfactry predictions for other
loading responses. The representation (5) should therefore
be correlated approximately with both constant stress and
constant strain-rate responses in some optimal manner, but no
direct scheme has yet been devised.

The above results suggest that the constant strain-rate responses of ice may provide a better basis for a single integral representation. In the stress formulation (4), constant strain-rate implies a varying argument $\underset{\sim}{E}(\tau)$ in the kernel $\dot{\underset{\sim}{G}}[\underset{\sim}{E}(\tau), t-\tau]$, and no unique correlation with the function $\underset{\sim}{G}[\underset{\sim}{E}, t]$; only constant strain response determines $\underset{\sim}{G}[\underset{\sim}{E}, t]$. Alternately, in the strain formulation (5), a non-uniform response $\hat{\underset{\sim}{\sigma}}(\tau)$ to constant strain-rate in the kernel $\dot{\underset{\sim}{J}}[\hat{\underset{\sim}{\sigma}}(\tau), t-\tau]$ yields a sequence of integral equations for $\underset{\sim}{J}$ with no unique solution. The same situation arises if constant strain is prescribed in the representation (5), discussed in more detail by Morland and Spring (1982), which shows that $\underset{\sim}{G}$ does not determine $\underset{\sim}{J}$, and similarly $\underset{\sim}{J}$ does not determine $\underset{\sim}{G}$. That is, the single integral stress and strain formulations (4) and (5) are not equivalent, in contrast to the relaxation and creep integral relations of linear viscoelasticity for which the relaxation and creep functions (of time only) are related uniquely by a linear Volterra integral equation. Hence, a kernel which can be determined by constant strain-rate response must depend only on strain-rate history and time, so we now examine two such integral representations of the stress formulation (1).

The direct analogy to the representation (4) is

$$\hat{\underset{\sim}{\sigma}}(t) = \underset{\sim}{T}_S[\underset{\sim}{E}'(t), 0] + \int_{-\infty}^{t} \dot{\underset{\sim}{T}}_S[\underset{\sim}{E}'(\tau), t-\tau]d\tau , \qquad (9)$$

with

$$\underset{\sim}{T}_S(\underset{\sim}{0}, t) \equiv 0 , \qquad (10)$$

so that the response to constant strain-rate $\underset{\sim}{E}'(\tau) = \underset{\sim}{r}_s H(\tau)$
is

$$\underset{\sim}{E}'(\tau) = \underset{\sim}{r}_s H(\tau): \qquad \hat{\underset{\sim}{\sigma}}(t) = \underset{\sim}{T}_s[\underset{\sim}{r}_s, t] \quad , \tag{11}$$

which determines the kernel $\underset{\sim}{T}_s[\underset{\sim}{r}_s, t]$ over the range of $\underset{\sim}{r}_s$
covered. Since $\underset{\sim}{E}'(t)$ is a rate of strain relative to a
given reference configuration, the representation (9) defines
a viscoelastic solid, and since bounded argument $\underset{\sim}{E}'(t)$ is
implied, no strain jumps arise; a stress jump induces a jump
in strain-rate. This is a satisfactory approximation for ice
in many situations. Also, anisotropy of the reference
configuration can be described through the structure of the
tensor kernel $\underset{\sim}{T}_s[\underset{\sim}{E}', t]$.

An alternative viscoelastic fluid relation can be constructed
from Noll's reduced form of the representation (1) when
dependence on the strain $\underset{\sim}{E}$ and rotation $\underset{\sim}{R}$ from a given
reference configuration is eliminated, namely

$$\underset{\sim}{\sigma}(t) = \underset{\sim}{\mathcal{G}}[\underset{\sim}{E}_t(\tau)]_{-\infty}^{t} \quad , \tag{12}$$

where $\mathcal{G}$ is an isotropic tensor function of the tensor $\underset{\sim}{E}_t$ to
satisfy frame indifference. Here $\underset{\sim}{E}_t(\tau)$ is the strain at time
τ relative to the configuration at time t, and is independent
of any given reference configuration. All configurations are
necessarily isotropic. $\underset{\sim}{E}_t'(\tau)$ is the strain-rate at time τ
relative to the current configuration. The appropriate integral
representation is

$$\underset{\sim}{\sigma}(t) = \underset{\sim}{T}_f[\underset{\sim}{E}_t'(t),0] + \int_{-\infty}^{t} \dot{\underset{\sim}{T}}_f[\underset{\sim}{E}_t'(\tau),t-\tau]d\tau \quad , \tag{13}$$

which satisfies the frame indifferent requirement (isotropy) if $\underset{\sim}{T}_f$ has a Rivlin-Ericksen expansion

$$\underset{\sim}{T}_f[\underset{\sim}{E}_t'(\tau),t] = \phi_0\underset{\sim}{1} + \phi_1\underset{\sim}{E}_t'(\tau) + \phi_2\{\underset{\sim}{E}_t'(\tau)\}^2 \tag{14}$$

where the response coefficients ϕ_α $(\alpha = 0,1,2)$ are functions of the invariants of $\underset{\sim}{E}_t'(\tau)$ and time. Also

$$\underset{\sim}{T}_f(\underset{\sim}{0},t) \equiv 0 \iff \phi_0 = 0 \quad \text{when} \quad \underset{\sim}{E}_t'(\tau) = \underset{\sim}{0} . \tag{15}$$

Thus $\underset{\sim}{T}_f$ is a polynominal in the tensor $\underset{\sim}{E}_t'(\tau)$, but has arbitrary dependence on its invariants. If the solid defined by the representation (9) is isotropic in the reference configuration, then $\underset{\sim}{E}$ is replaced by $\bar{\underset{\sim}{E}} = \frac{1}{2}(\underset{\sim}{F}\underset{\sim}{F}^T - \underset{\sim}{1})$, $\hat{\underset{\sim}{\sigma}}$ by $\underset{\sim}{\sigma}$, and $\underset{\sim}{T}_s$ has an expansion in $\bar{\underset{\sim}{E}}'$ analogous to the expansion (14) of $\underset{\sim}{T}_f$ in $\underset{\sim}{E}_t'$. The response to constant strain-rate $\underset{\sim}{E}_t'(\tau) = \underset{\sim}{r}_f H(\tau)$ given by the representation (13) is

$$\underset{\sim}{E}_t'(\tau) = \underset{\sim}{r}_f H(\tau): \qquad \underset{\sim}{\sigma}(t) = \underset{\sim}{T}_f[\underset{\sim}{r}_f,t] \tag{16}$$

which determines the kernel $\underset{\sim}{T}_f[\underset{\sim}{r}_f,t]$ over the range of $\underset{\sim}{r}_f$ covered.

While the strain-rate $\underset{\sim}{D}(t)$ relative to the current configuration, defined by the symmetric part of the spatial velocity gradient at time t, is simply

$$\underset{\sim}{D}(t) = \underset{\sim}{E}_t'(t) , \tag{17}$$

a kernel dependence on $\underset{\sim}{D}(\tau)$ would require rates measured with respect to the continuously changing sequence of configurations as the integration variable τ changes, and does not appear to be a natural, or practical, representation. Also the fluid representation (13), (14) is distinct from the isotropic form of the solid relation (9), since the relative strain-rate $\underset{\sim}{E}_t'(\tau)$ and strain-rate $\overline{\underset{\sim}{E}}'(\tau)$ from a reference configuration are not identical. However, for small strain $||\underset{\sim}{E}|| << 1$ and small rotation $||\underset{\sim}{R} - \underset{\sim}{1}|| << 1$, if terms of these magnitudes are neglected in comparison with unity, then

$$\underset{\sim}{E}'(\tau) = \overline{\underset{\sim}{E}}'(\tau) = \underset{\sim}{E}_t'(\tau) \ . \tag{18}$$

In this situation the constant strain-rates $\underset{\sim}{r}_s$ and $\underset{\sim}{r}_f$ in the results (11) and (16) are the same, and the kernels $\underset{\sim}{T}_s$ and $\underset{\sim}{T}_f$ are the same, so the representations (9) and (13) are equivalent. If though a common function $\underset{\sim}{T}$ is adopted for $\underset{\sim}{T}_s$ and $\underset{\sim}{T}_f$ and extended to large strain, with small or large rotation, the respective arguments $\underset{\sim}{E}'(\tau)$ and $\underset{\sim}{E}_t'(\tau)$ become different and the representations (9) and (13) are distinct. We will adopt the common representation for small strain and zero rotation to investigate the uni-axial stress description determined by a typical response illustrated in Fig. 1c.

3. Uni-axial stress relations

For uni-axial compressive stress σ, axial compressive engineering strain e and equal lateral compressive engineering strains e_2, zero rotation, and the incompressibility

approximation,

$$\sigma_{11} = -\sigma \ , \quad \sigma_{ij} = 0 \quad (i \neq 1, \ j \neq 1),$$

$$E_{11} = -e(1 - \tfrac{1}{2}e), \quad E_{22} = E_{33} = \tfrac{1}{2}e(1 - e)^{-1} . \tag{19}$$

In the small strain approximation $E_{11} = -e$, both representations (9) and (13) yield an axial relation

$$\sigma(t) = T[e'(t),0] + \int_{-\infty}^{t} \dot{T}[e'(\tau),t-\tau]d\tau \ , \tag{20}$$

where $T[r,t]$ is the stress $\sigma(t)$ required to maintain a constant strain-rate $e(t) = rH(t)$. Later calculations will use functions $T[r,t]$ constructed from families of stress responses for different r with the shape illustrated in Fig. 1c. Given the kernel $T(r,t)$ and a prescribed stress history $\sigma(t)$, the relation (20) is a non-linear Volterra integral equation for the strain-response $e(t)$. Incompressibility implies that the fluid relation (13) and isotropic form of the solid relation (9) have only two independent principal deviatoric components, since the deviatoric kernel $\underset{\sim}{T}$ satisfies $\mathrm{tr}\,\underset{\sim}{T} = 0$, but the uni-axial configuration (19) yields only one independent component with

$$T_{22} = -\frac{1}{2} T_{11} \ , \quad T_{11} = -\frac{2}{3} T. \tag{21}$$

Thus tests with two independent stress components are required to determine the deviatoric tensor relation.

Let a constant stress $\sigma(t) = \sigma_c H(t)$ be prescribed in the integral equation (20), so $e(t) = 0$ for $t \leq 0$.

Define

$$t_i = i\delta \quad , \quad r_i = e'(t_i), \quad i = 0,1,2,\dots \ , \qquad (22)$$

where δ is a small time interval, and interpret r_o as the strain-rate at $t = 0+$; that is, the strain-rate jump at $t = 0$. By equation (20),

$$\bullet \qquad \sigma_c = T[r_o, 0], \qquad\qquad (23)$$

which has a unique solution $r_o(\sigma_c)$ if $T[r,0]$ is monotonic increasing in r; that is, if constant strain-rate $rH(t)$ requires an initial stress jump σ which increases with r, assumed here. This stress jump, instead of the smooth rapid rise from zero illustrated in Fig. 1c, is a necessary feature of representations with no strain dependence, and will be incorporated in the model functions. Now apply the trapezoidal rule to the first time interval of equation (20), then

$$\sigma_c = T[r_1, 0] + \tfrac{1}{2}\delta\{\dot{T}[r_1,0] + \dot{T}[r_o,\delta]\} \qquad (24)$$

which is an implicit equation for r_1. Again, assuming that $\dot{T}[r,0]$ is monotonic increasing in r; that is, the initial stress-rate at constant strain-rate r increases with r, there is a unique solution $r_1(\sigma_c)$. Continued application of the trapezoidal rule over successive time intervals gives for $n = 2,3,\dots$

$$\sigma_c = T[r_n, 0] + \tfrac{1}{2}\delta\dot{T}[r_n, 0]$$

$$+ \delta\{\tfrac{1}{2}\dot{T}[r_o, t_n] + \sum_{i=1}^{n-1} \dot{T}[r_i, t_{n-1}]\}, \qquad (25)$$

which is an implicit equation for r_n once
r_i $(i = 0,1,\ldots,n-1)$ are determined, with unique solution
by the above monotonicity properties.

Although the algorithm (23) - (25) yields a sequence
of implicit equations for each r_n $(n = 1,2,\ldots)$, the
unknown r_n occurs only in a monotonic function comprising
two terms, and numerical solution is simple and accurate.
Tests of accuracy, and the main solution features, can be made
by comparison with exact solutions for an associated linear
viscoelastic relation given by taking

$$T[e',t] \equiv T_\ell(t)e' , \quad T[r_i,t_j] = T_\ell(t_j)r_i , \qquad (26)$$

when the axial relation (20) becomes

$$\sigma(t) = T_\ell(0)e'(t) + \int_{-\infty}^{t} \dot{T}_\ell(t-\tau)e'(\tau)d\tau . \qquad (27)$$

Let the Laplace transform of $f(t)$ defined on $t \geq 0$ be
denoted by $L[f(t)] = \overline{f}(s)$, and define $\tilde{f}(s) = s\overline{f}(s)$, then
for strain and stress histories zero on $t \leq 0$, the transform
of the convolution relation (27) gives

$$\overline{e'}(s) = \overline{\sigma}(s)/\tilde{T}_\ell(s) . \qquad (28)$$

For the illustrations, $T_\ell(t)$ is represented by a series of
exponentials, and for $\overline{\sigma}(s) = \sigma_c/s$, $\overline{e'}(s)$ can then be inverted
analytically to express $e'(t)$ as a series of exponentials.

4.   Illustrations

The constant strain-rate response illustrated in Fig. 1c,
but with an initial non-zero stress, has been modelled by the

kernel functions

$$T[r,t] = \sigma_M(r)\{A + Be^{-brt} - De^{-drt}\} \tag{29}$$

for three different sets of constants A, B, D, b, d, with $d > b > 0$. Since t enters only in the product $e = rt$, the maximum stress for each r occurs at a strain e_M given by

$$\exp\{(d-b)e_M\} = dD/bB, \tag{30}$$

which is indpendent of r, consistent with observations. We adopt the value $e_M = 0 \cdot 01$, then for maximum stress $\sigma_M(r)$,

$$A + Be^{-0 \cdot 01b} - De^{-0 \cdot 01d} = 1. \tag{31}$$

The initial stress is

$$T[r,0] = \sigma_M(r)\{A + B - D\}, \tag{32}$$

which requires

$$A + B - D > 0, \tag{33}$$

and a bounded positive long time (large strain) limit $A\sigma_M(r) < \sigma_M(r)$ requires

$$1 > A > 0. \tag{34}$$

Also, the strain e_I $(> e_M)$ at inflexion, $\ddot{T} = 0$, is given by

$$\exp\{(d-b)e_I\} = d^2D/b^2B. \tag{35}$$

We prescribe A, d/b, e_I/e_M with $e_M = 0 \cdot 01$ to calculate the three sets of constants shown in Table 1 from the relations (30), (31), (35).

	I	II	III
e_I/e_M	1·75	1·84	1·90
A	0·7	0·7	0·7
B	0·6412	0·4520	0·3191
D	1·0965	0·7009	0·5323
b	53·6479	30·4575	5·1685
d/b	5	10	100

Table 1. Parameters for three families of constant
strain-rate responses

Figure 2 shows the corresponding curves for $T[r,t]/\sigma_M(r) = \sigma/\sigma_M$ as functions of $e = rt$. Following Mellor's (1980) conjecture, the maximum stress relation $\sigma_M(r)$ is assumed to be the inverse function of the minimum strain-rate $r_m(\sigma)$ at constant stress, and we adopt a close least squares fit to laboratory data at 273·13K over the axial stress range $0 \le \sigma \lesssim 9 \times 10^5 Nm^{-2}$ derived by Smith and Morland (1981):

$$\sigma_M(r) = 1 \cdot 991 \tan^{-1}(1 \cdot 609r) + 4 \cdot 2929 \tan^{-1}(0 \cdot 0206r), \qquad (36)$$

where σ is measured in units $10^5 Nm^{-2}$ and r in units a^{-1}. $\sigma_M'(r) > 0$ so $T[r,0]$ is monotonic increasing as required, and further, $\dot{T}[r,0]$ is monotonic increasing as required since

$dD > bB$ by equation (30) with $d > b$.

A comparison linear viscoelastic relation (27) is given
by

$$T_\ell(t) = k\{A + Be^{-bt} - De^{-dt}\},\tag{37}$$

where k is a constant. Figure 3 shows the stress
$\sigma = T_\ell(t)r$ as a function of $e = rt$ for three values of
constant strain-rate r using the parameter set I in Table 1
and $k = 4 \cdot 5 \times 10^5 Nm^{-2}a^{-1}$. Here the maximum stress occurs
at constant $t_M = 0 \cdot 01a$ for all r , and hence at $e_M = rt_M$
which is proportional to r . The Laplace transform solution
(28) for $\sigma(t) = \sigma_c H(t)$ is

$$\overline{e'}(s) = \frac{\sigma_c(s+b)(s+d)}{ks\{A(s+b)(s+d) + Bs(s+d) - Ds(s+b)\}}$$

$$= \frac{\sigma_c}{k}\left\{\frac{U}{s} - \frac{V}{s+v} + \frac{W}{s+w}\right\},\tag{38}$$

giving

$$e'(t) = \frac{\sigma_c}{k}H(t)\left\{U - Ve^{-vt} + We^{-wt}\right\}.\tag{39}$$

With the above parameters and t in units a^{-1},

$$U = 1 \cdot 4286, V = 0 \cdot 5608, W = 3 \cdot 2183, v = 30 \cdot 428, w = 1352 \cdot 71,\tag{40}$$

and Fig. 4 shows the corresponding $ke'(t)/\sigma_c$. Comparison
values at different times, including the time $t_m(\sigma_c) = 36 \cdot 6\,hrs$
to minimum strain-rate, of the analytic solution $ke'(t)/\sigma_c$ and
that given by the numerical algorithm (23) - (25) applied to

the integral equation (27), are shown in Table 2. The numerical

t hrs	ke'(t)/σ_c(anal)	ke'(t)/σ_c(num)	error
0	4·0861	4·0861	0
10	1·5740	1·5739	-1×10^{-4}
36·6	0·9460	0·9460	0
100	1·0323	1·0323	0
1000	1·4112	1·4105	-7×10^{-4}

Table 2. Comparison of analytic and numerical strain-rates
at constant stress for linear viscoelastic relation.

algorithm is clearly accurate.

Applying the algorithm to the non-linear relation (29)
with parameter sets I, II, III in turn gives the respective
strain-rates at different constant stresses σ_c. The
minimum strain-rate $r_m(\sigma_c)$, time to minimum $t_m(\sigma_c)$, and strain
at minimum $e_m(\sigma_c)$, are given in Table 3. The strain $e_m(\sigma_c)$

	$\sigma_c 10^5 Nm^{-2}$	$r_m(\sigma_c)a^{-1}$	$t_m(\sigma_c)$hrs	$e_m(\sigma_c)$
I	0·25	0·0806	1048	0·0111
	0·5	0·1542	524	0·0114
	0·75	0·2540	348	0·0119
	1·0	0·3549	258	0·0128
	1·5	0·6193	165	0·0155
	2·0	1·0863	109	0·0196
II	0·5	0·1597	524	0·0107
	1·0	0·3429	254	0·0113
III	0·5	0·1560	628	0·0121
	1·0	0·3326	306	0·0127

Table 3. Constant stress results for three parameter sets.

is approximately $0 \cdot 01$ for all three sets for $\sigma_c \leq 10^5 \text{Nm}^{-2}$, but increases to approximately $0 \cdot 02$ at $\sigma_c = 2 \times 10^5 \text{Nm}^{-2}$. Figure 5 shows the normalised strain-rate $e'(t)/r_m(\sigma_c)$ as a function of $t/t_m(\sigma_c)$ for different values of σ_c. The response at $\sigma_c = 2 \times 10^5 \text{Nm}^{-2}$ is not satisfactory for long times, but the corresponding creep curves showing $e(t) = \int_0^t e'(\tau)d\tau$ in Fig. 6 indicate that the strain e is becoming large and outside the scope of the representation. The responses at low stresses are very similar, and are consistent with the required shapes in Figs 1b and 1a. Figures 7 and 8 compare the strain-rate and creep responses at $\sigma_c = 10^5 \text{Nm}^{-2}$ for the three parameter sets, showing that the response is not sensitive to kernel detail when the representation and kernel are determined by constant strain-rate.

5.   Conclusions

Single integral representations of non-linear viscoelastic response are determined fully by one type of test, but different forms of representation are required for different types of test, and they are not equivalent. The integral kernel is determined directly by the appropriate test response, in one, two, or three-dimensional tests, and provides a much simpler correlation with test data than the viscoelastic relations of differential type. An integral representation also offers numerical advantages in the solution of boundary-value problems. It has been shown that stress formulations for ice depending only on strain-rate history, not strain, of solid type and of

fluid type, which are determined by the stress history required to maintain constant strain-rate, can predict an appropriate shape of strain-rate response to constant uni-axial stress for small strain. A strain formulation based on constant stress response predicts less satisfactory response to constant strain-rate. An improved single integral approximation may be possible by correlating the above stress formulation approximately with constant strain-rate data within a range of kernel parameters, and seeking an optimum parameter set for approximate agreement with constant stress data.

Acknowledgement

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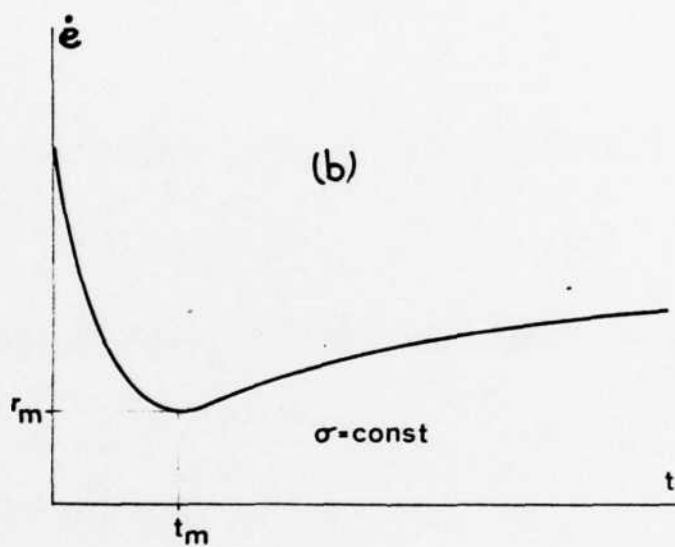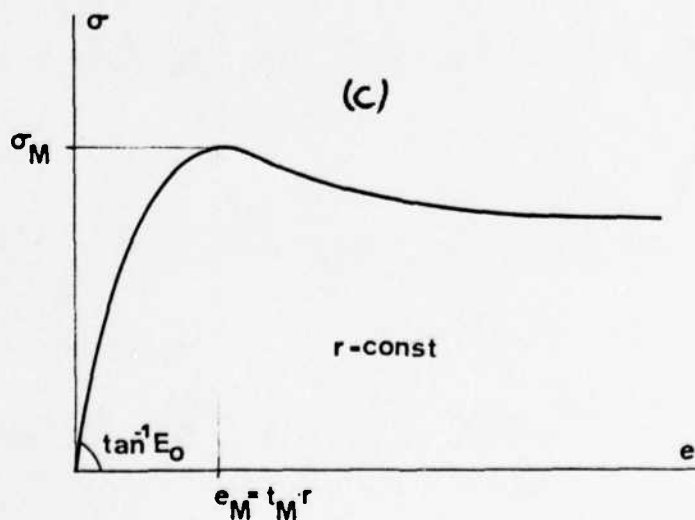Figure Captions

1. Typical responses of ice in uni-axial compressive
   stress σ :
   (a) strain $e(t)$ at constant stress, (b) strain-rate
   $\dot{e}(t)$ at constant stress, (c) stress $\sigma(t)$ at constant
   strain-rate.

2. Constant strain-rate responses for parameter sets
   I (——), II (----), and III (-·-·).

3. Linear viscoelastic stress response at constant strain-rate.

4. Linear viscoelastic strain-rate at constant stress.

5. Strain-rate response to constant stresses for parameter
   set I: $\sigma_c = 0 \cdot 25$ (-..-..), 1 (——), $1 \cdot 5$ (----),
   2(-·-·) $\times 10^5 \mathrm{Nm}^{-2}$.

6. Strain response to constant stresses for parameter set
   I: $\sigma_c = 0 \cdot 25$ (-..-..), 1(——), $1 \cdot 5$ (----), 2(-·-·) $\times 10^5 \mathrm{Nm}^{-2}$.

7. Strain-rate response to constant stress $\sigma_c = 10^5 \mathrm{Nm}^{-2}$
   for parameter sets I (——), II (----), and III (-·-·).

8. Strain response to constant stress $\sigma_c = 10^5 \mathrm{Nm}^{-2}$ for
   parameter sets I (——), II (----), and III (-·-·).
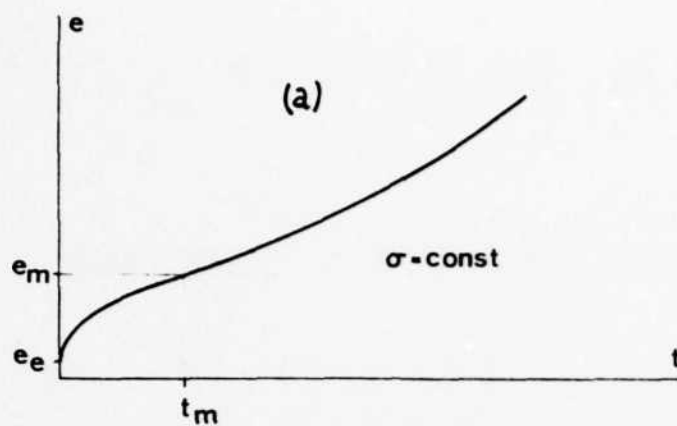
Figure 1. Typical responses of ice in uni-axial compressive
stress σ: (a) strain $e(t)$ at constant stress,
(b) strain-rate $\dot{e}(t)$ at constant stress, (c)

Figure 2. Constant strain-rate responses for parameter sets
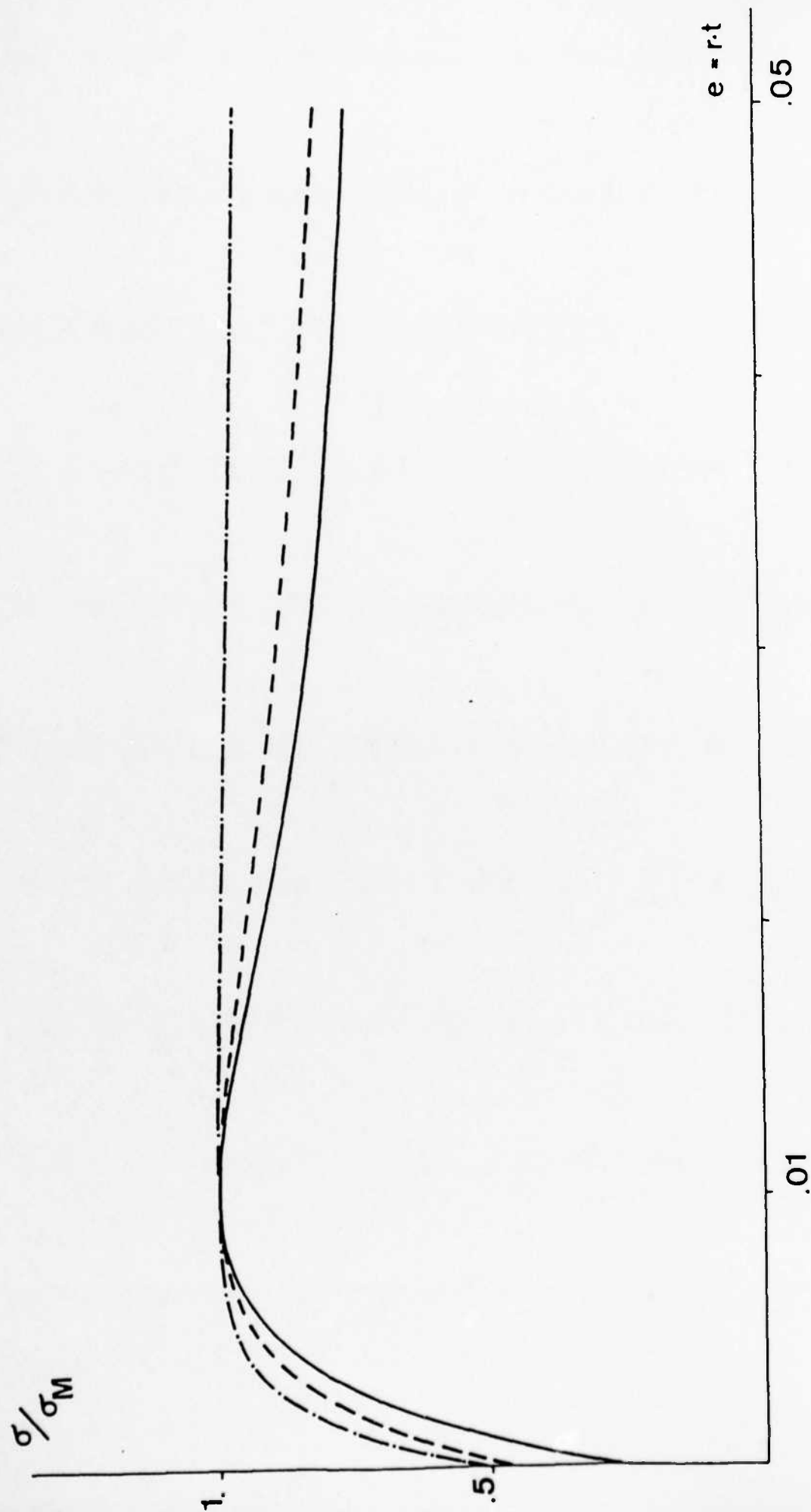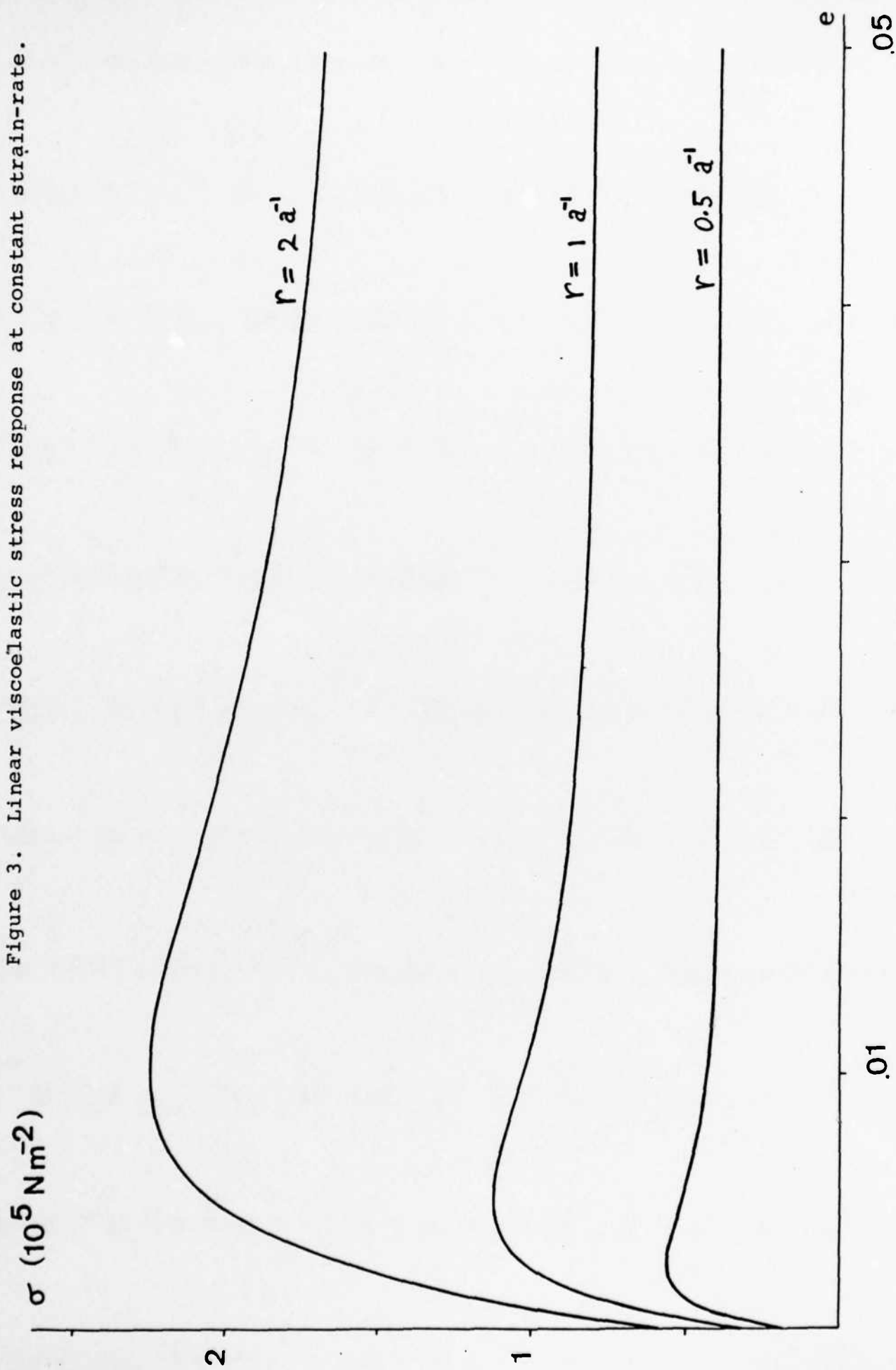
I (——), II (–·–·–), and III (–··–··).

Figure 3. Linear viscoelastic stress response at constant strain-rate.

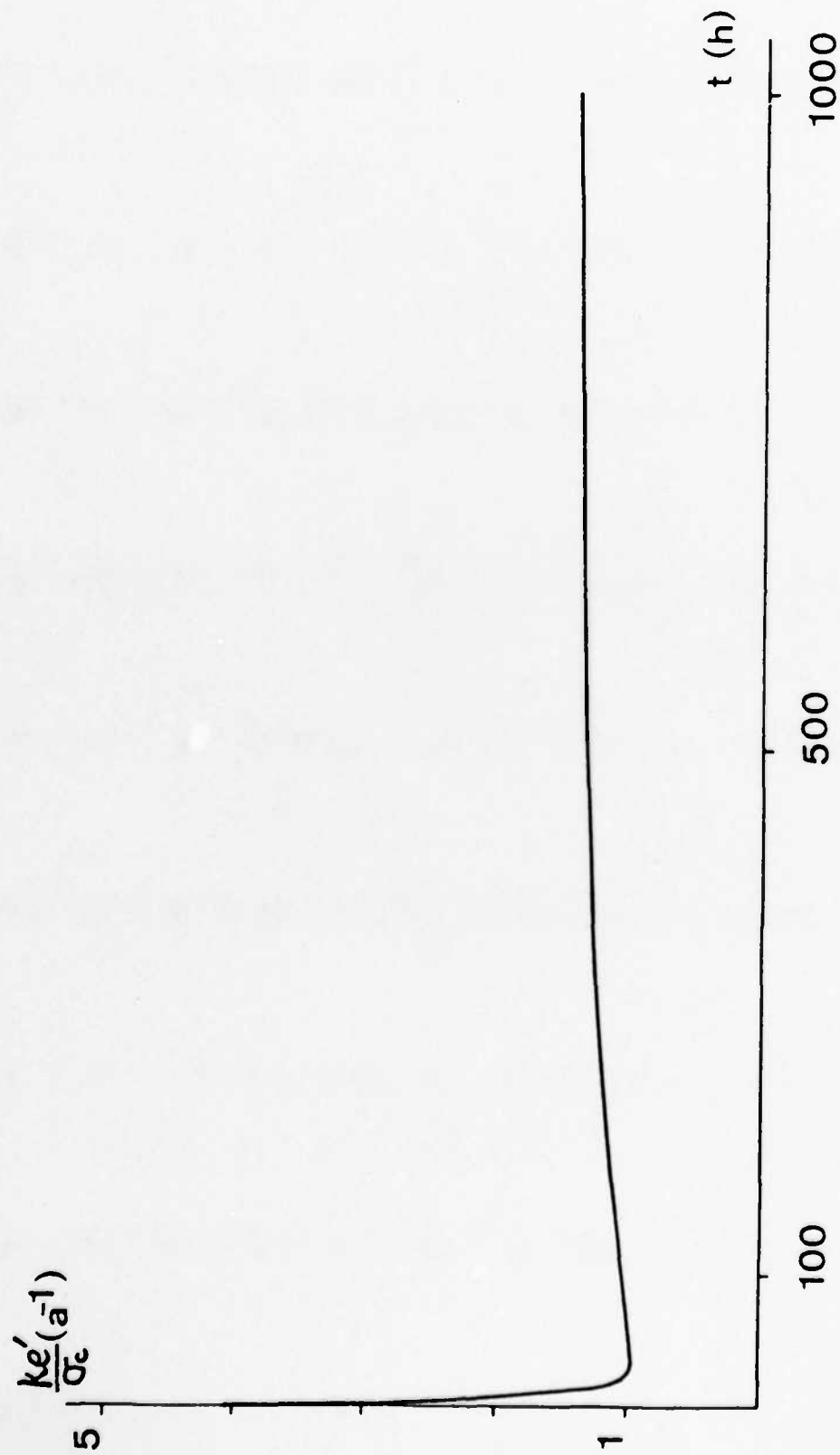Figure 4. Linear viscoelastic strain-rate at constant stress.

Figure 5. Strain-rate response to constant stresses for parameter set I: $\sigma_c = 0.25$ (-··-··), 1 (———), 1·5 (———), 2 (-·-·) $\times 10^5 \text{Nm}^{-2}$.
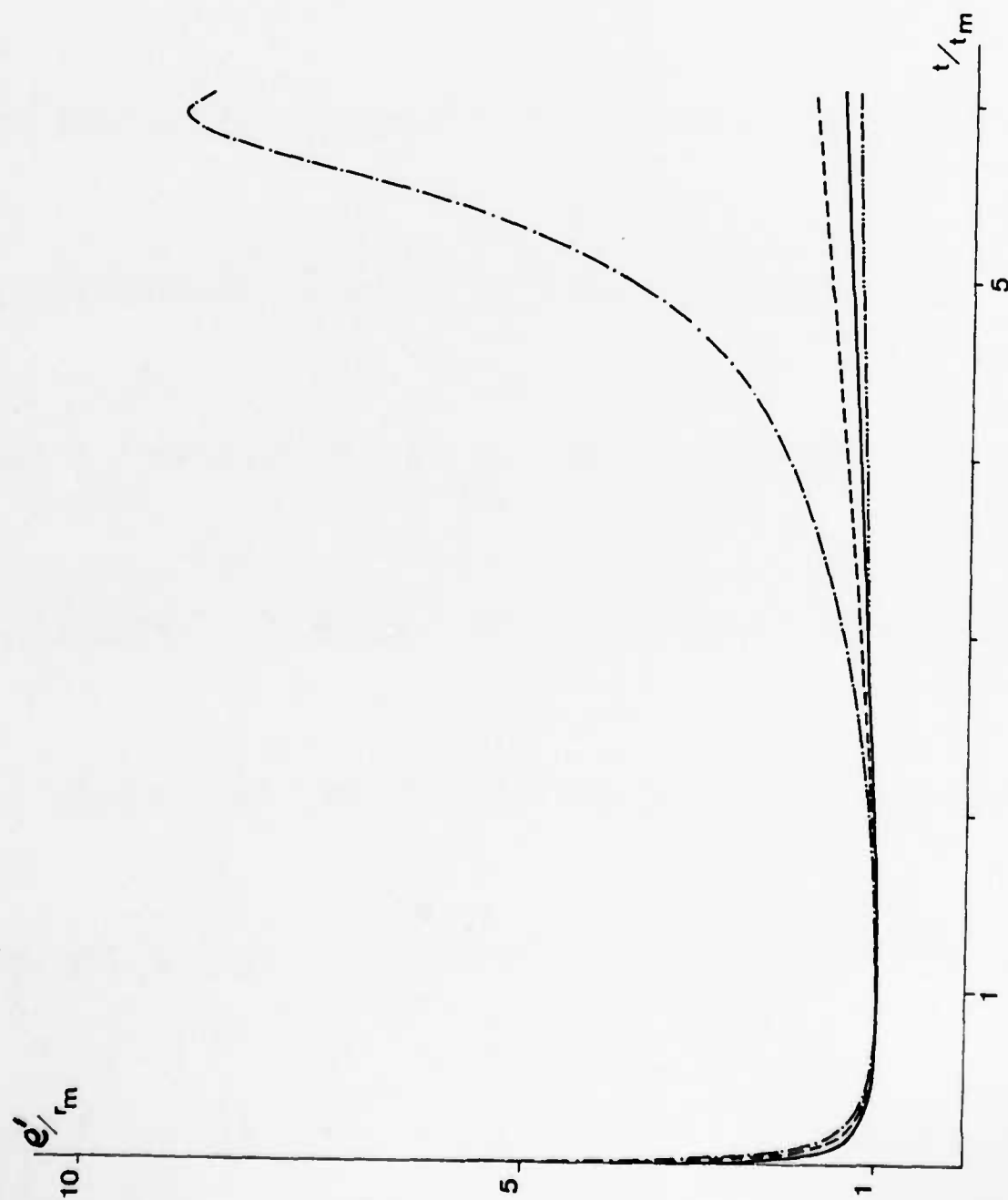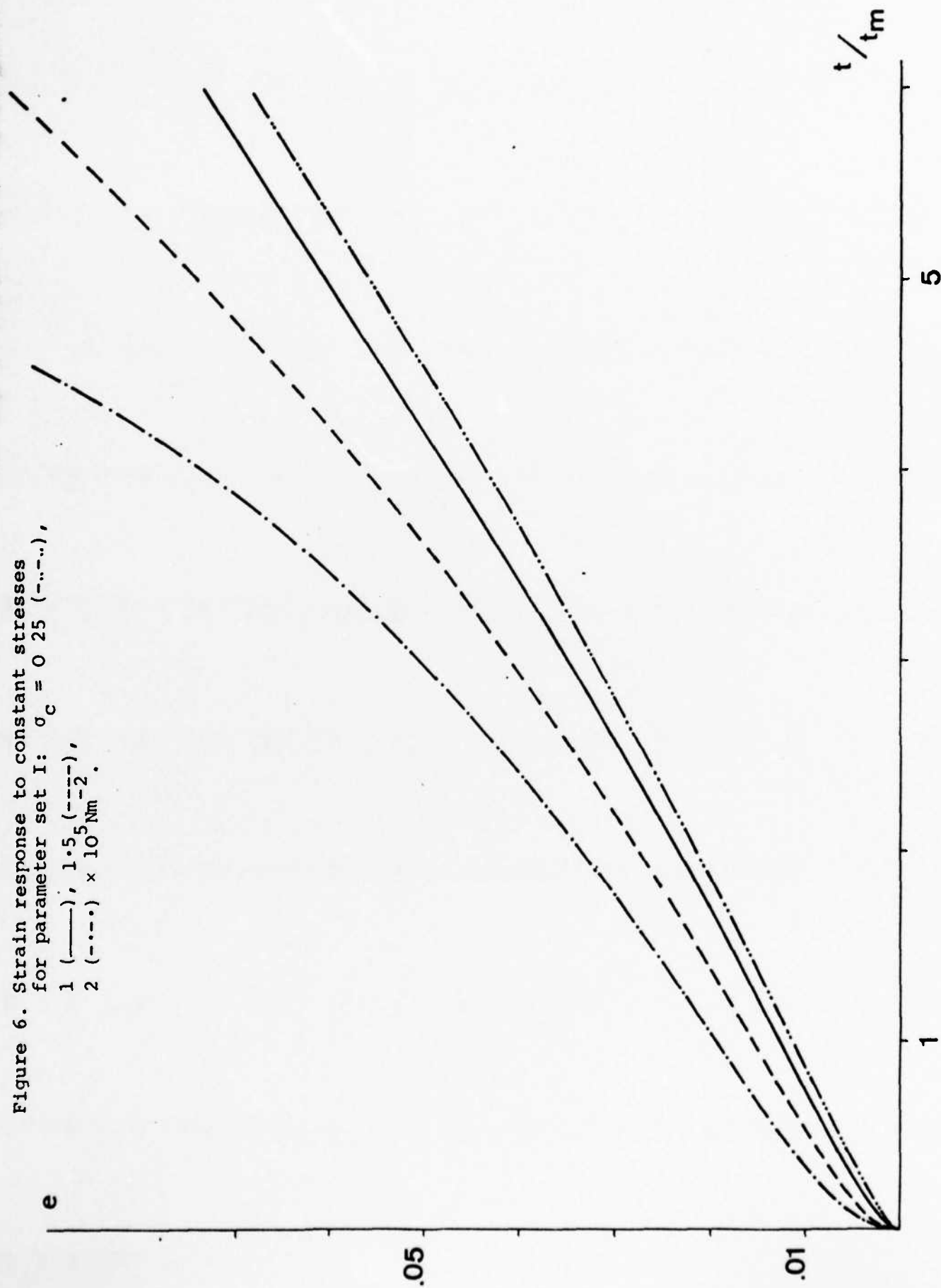
Figure 6. Strain response to constant stresses for parameter set I: σ_c = 0 25 (-··-··),
1 (_____), 1·5 (-----),
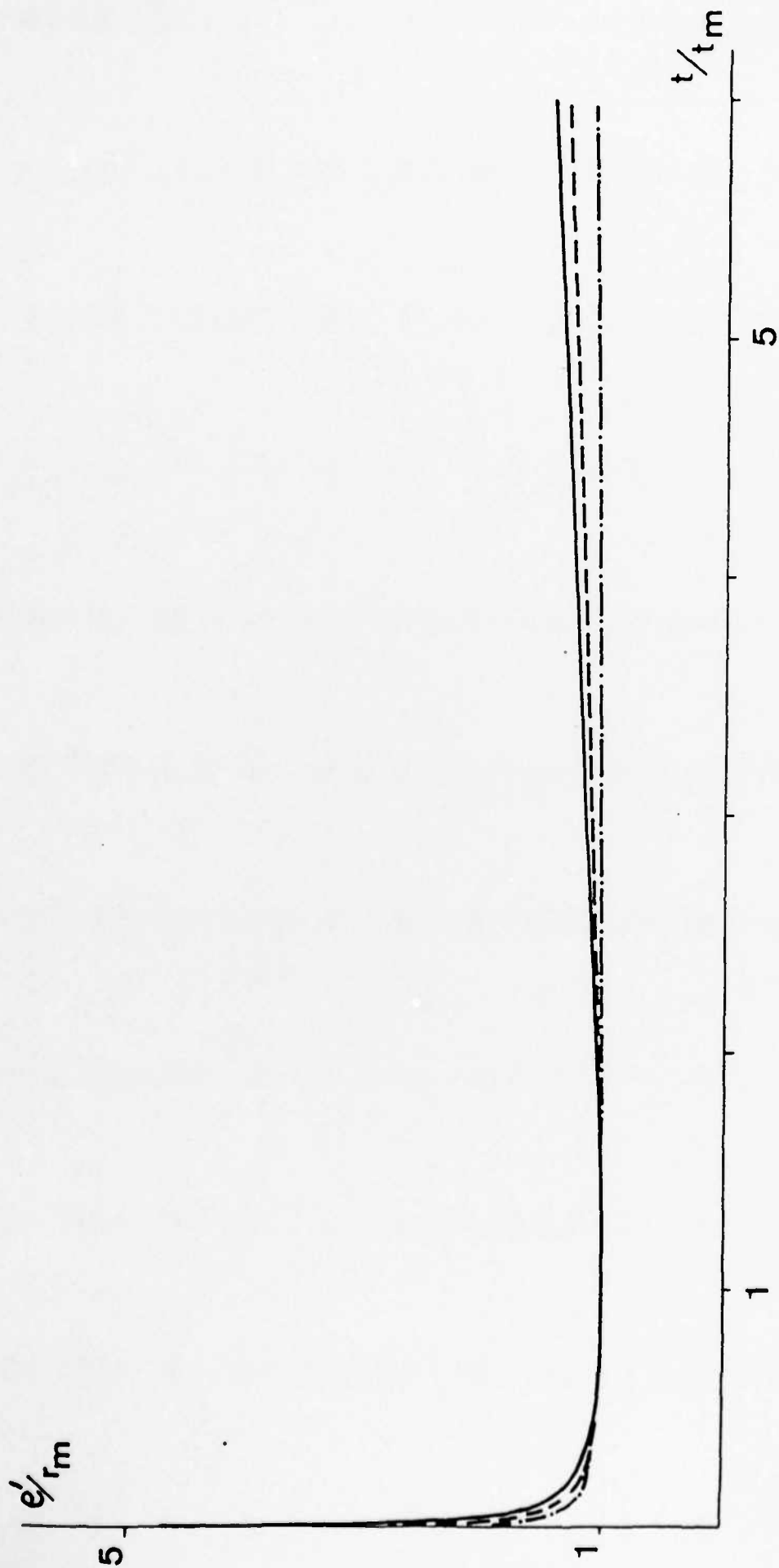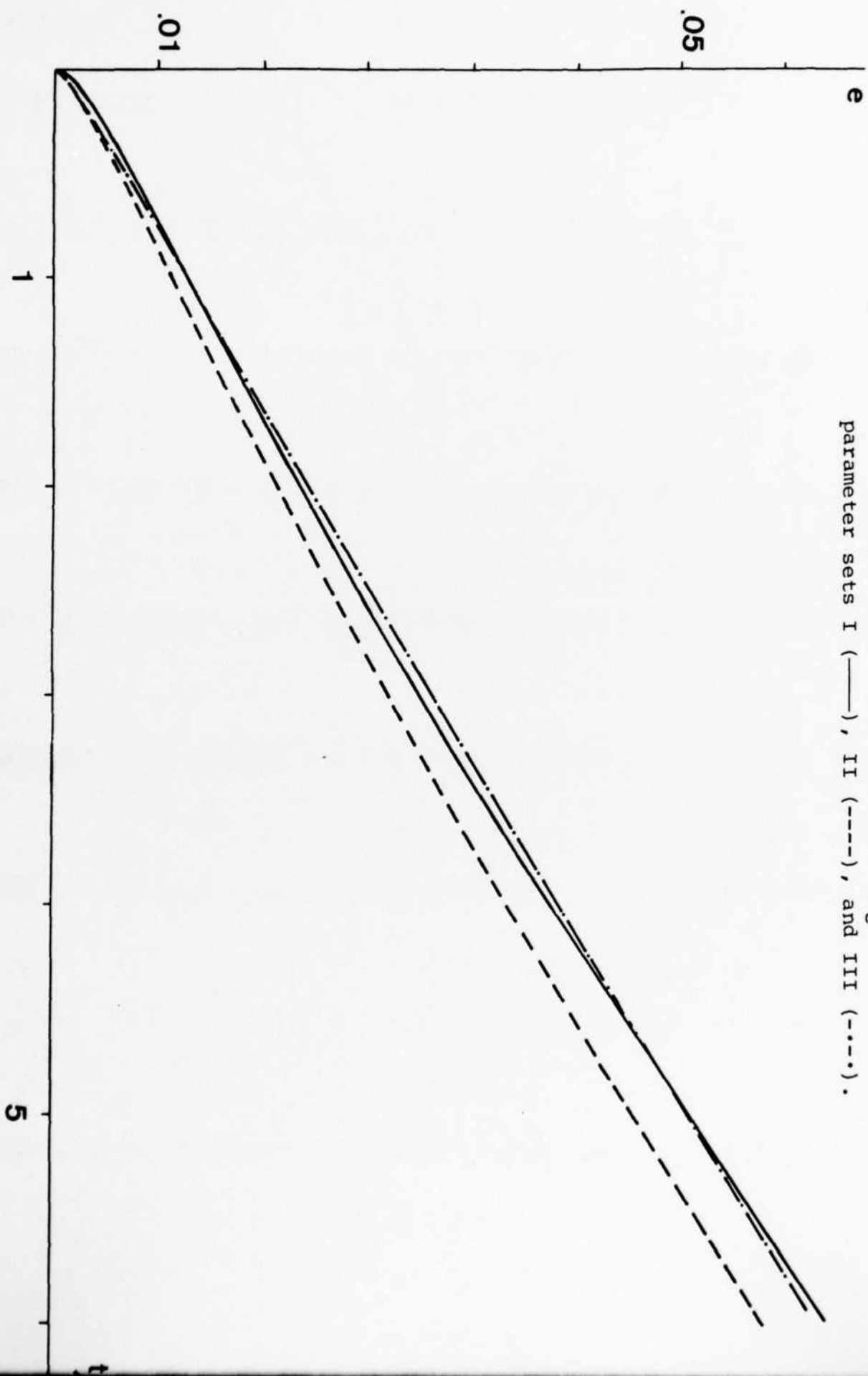2 (-·-·-·) $\times 10^5 \, \mathrm{Nm}^{-2}$.

Figure 7. Strain-rate response to constant stress $\sigma_c = 10^5 \text{Nm}^{-2}$ for parameter sets I (———), II (———), and III (—·—·).

Figure 8. Strain response to constant stress $\sigma_c = 10^5 Nm^{-2}$ for parameter sets I (———), II (———), and III (—··—).

PART II

SINGLE INTEGRAL REPRESENTATIONS IN ICE MECHANICS

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Abstract

A single integral viscoelastic constitutive equation for ice is developed which possesses significant theoretical and practical advantages over previously suggested equations of this type (Spring and Morland, 1983). The theory is specialised to the case of small strain uniaxial compression and the resulting constitutive equation is shown to verify the relations between the experimental data obtained in constant load (CL) creep tests and constant displacement rate (CD) "strength" tests conjectured in Mellor and Cole (1982) and demonstrated in Mellor and Cole (1983).

1. Introduction

Constant strain-rate (CD) uniaxial compression tests performed on ice exhibit highly anomalous physical response in unusual and striking ways. A typical response curve is given, for the purpose of illustration, in figure 1. It can be seen from the graph that the stress-strain curve is highly non-linear and, since the strains are of the order of one per cent, conventional theories of non-linear viscoelasticity (such as finite linear viscoelasticity), which attribute the non-linearity to large strains, cannot be expected to apply. The variation of the peak stress or strength σ_{max} with the constant strain-rate is also non-linear.

The most striking feature, however, of the curve is its non-monotonic nature, i.e. the fact that the stress increases to a maximum value at the failure strain ϵ_f (typically around 1%) and then decays to an approximately constant long-time value. It is demonstrated later that the theory of linear viscoelasticity is incapable of giving a physically satisfactory account of this feature. The single integral representation of Spring and Morland (1983) also fails in this respect but the approach adopted here shows how their ideas can be modified to provide a satisfactory model of this non-monotonicity.

The proposed model of ice given here is developed by specialising a non-linear single integral representation for the stress, which is valid for arbitrary strains, to the case of non-linear response to small strains. The constitutive equation adopted for the starting point in this process is of the fluid

type, and presupposes that the material under consideration is mechanically incompressible (a reasonable approximation for ice).

The three dimensional small strain representation thus obtained is then further specialised to the case of uniaxial compression, and three types of uniaxial compression test are then considered. These tests are the constant strain (stress relaxation) test, the constant strain-rate test and the constant stress (creep) test. The former test is not significant experimentally but is theoretically important because it imposes physical conditions on the model which must be met.

Exact solutions are obtained for the constant strain and constant strain-rate tests but the solution of the constant stress test involves a non-linear Volterra integral equation of the first kind, which presents serious difficulty both numerically and analytically. This difficulty can, however, be circumvented by means of an approximate solution, which bears out the conjectures of Mellor and Cole (1982, Table 1, p. 206) on the correspondence between constant strain-rate and constant stress data.

To conclude, a possible empirical form of the kernel of the single integral law is examined.

2. The Constitutive Equation

As a starting point consider the reformulation

$$\underset{\sim}{\sigma}(t) = \underset{\sim}{\sigma}\overset{t}{\underset{-\infty}{}}[J_t(\tau)] \tag{1}$$

of the constitutive equation given for the Cauchy stress in a viscoelastic fluid by Spring and Morland (1983, eq. (12)), where

$$\underset{\sim}{J}_t(\tau) = \underset{\sim}{F}(t)\underset{\sim}{F}^{-1}(\tau)\{\underset{\sim}{F}^{-1}(\tau)\}^T \underset{\sim}{F}^T(t) \tag{2}$$

is the Finger strain tensor(Lodge, 1974, p.105) and $\underset{\sim}{F}$ is the deformation gradient tensor. Following Morland and Spring (1982) suppose that the constitutive equation (1) can be expressed as a single integral representation

$$\underset{\sim}{\sigma}(t) = \int_{-\infty}^{t} \underset{\sim}{f}[J_t(\tau), t - \tau]d\tau \quad . \tag{3}$$

While the single integral formulation (3) appears different to that adopted by Morland and Spring (1982), it can be shown that the two formulations are equivalent.

In order to make further progress it is necessary to make simplifying assumptions. Suppose that the material is mechanically incompressible, a conventional and satisfactory

approximation which leads to the constraints

$$\det \underset{\sim}{J}_t(\tau) = 1 \quad , \quad \det \underset{\sim}{F} = 1 \quad , \tag{4}$$

and requires that the constitutive equation (3) only determines the Cauchy stress up to an arbitrary isotropic pressure p, where

$$\underset{\sim}{\sigma}(t) = - p\underset{\sim}{I} + \underset{\sim}{\sigma}_E(t) \quad . \tag{5}$$

$\underset{\sim}{\sigma}_E$ is known as the extra stress. It should be noted that the decomposition (5) is not unique but is commonly made so by taking $\underset{\sim}{\sigma}_E$ to be the deviatoric stress: this is not adopted here, however, because it leads to an inconvenient form for the reduced constitutive equation derived later.

If the constitutive equation (3) is to provide a satisfactory basis for a physical theory of viscoelastic materials it must comply with the principle of material frame indifference (objectivity). This requirement, together with the restrictions (4) and (5) imposed by incompressibility, leads to the Rivlin-Ericksen representation

$$\underset{\sim}{\sigma}(t) = - p\underset{\sim}{I} + \int_{-\infty}^{t} f_1(I_{\underset{\sim}{J}}, II_{\underset{\sim}{J}}, \ t - \tau) \underset{\sim}{J}_t(\tau) d\tau$$

$$+ \int_{-\infty}^{t} f_{-1}(I_{\underset{\sim}{J}}, II_{\underset{\sim}{J}}, \ t - \tau) \underset{\sim}{J}_t^{-1}(\tau) d\tau \quad , \tag{6}$$

where

$$I_{\underset{\sim}{J}} = \text{tr } \underset{\sim}{J}_t(\tau) \qquad \text{and} \qquad II_{\underset{\sim}{J}} = \text{tr } \underset{\sim}{J}_t^{-1}(\tau) \tag{7}$$

are the principal invariants of $\underset{\sim}{J}_t(\tau)$.

Now consider material response in a multiaxial constant strain test specified by the deformation gradient history

$$\underset{\sim}{F}(t) = \underset{\sim}{I} \qquad \text{whenever } t < 0$$
$$\underset{\sim}{F}(t) = \overline{\underset{\sim}{F}} \qquad \text{whenever } t > 0 \tag{8}$$

The substitution of the history (8) into the constitutive equation (6) via the relations (2) and (7) furnishes the result

$$\underset{\sim}{\sigma}(t) = -p\underset{\sim}{I} + \mu_1(I_{\underset{\sim}{B}}, II_{\underset{\sim}{B}}, t)\underset{\sim}{B}$$
$$+ \mu_{-1}(I_{\underset{\sim}{B}}, II_{\underset{\sim}{B}}, t)\underset{\sim}{B}^{-1} , \quad t > 0 , \tag{9}$$

where

$$\mu_\Gamma(I_{\underset{\sim}{B}}, II_{\underset{\sim}{B}}, t) = \int_t^\infty f_\Gamma(I_{\underset{\sim}{B}}, II_{\underset{\sim}{B}}, \tau)d\tau , \quad \Gamma = \pm 1, \tag{10}$$

$$\underset{\sim}{B} = \overline{\underset{\sim}{F}}\,\overline{\underset{\sim}{F}}^T , \tag{11}$$

$$I_{\underset{\sim}{B}} = \operatorname{tr} \underset{\sim}{B} \quad \text{and} \quad II_{\underset{\sim}{B}} = \operatorname{tr} \underset{\sim}{B}^{-1} \quad . \tag{12}$$

It should be noted that, in equation (9), a term given by

$$\int_o^t f_1(3, 3, t - \tau) \underset{\sim}{I} \, d\tau + \int_o^t f_{-1}(3, 3, t - \tau) \underset{\sim}{I} \, d\tau$$

has been incorporated into the arbitrary pressure p .

The comparison of equation (9) with the well-known constitutive equation

$$\underset{\sim}{\sigma} = - p\underset{\sim}{I} + \mu_1(I_{\underset{\sim}{B}}, II_{\underset{\sim}{B}})\underset{\sim}{B} + \mu_{-1}(I_{\underset{\sim}{B}}, II_{\underset{\sim}{B}})\underset{\sim}{B}^{-1} \tag{13}$$

of an incompressible isotropic elastic material provides motivation for further model restrictions. A particularly simple version of equation (13) is the Neo-Hookean constitutive equation obtained by setting $\mu_{-1} = 0$ and taking μ_1 to be a constant. The corresponding version of equation (9) is obtained in a similar manner with the exception that μ_1 must now be taken to be a function of t alone. It follows that, for the "Neo-Hookean" viscoelastic fluid, the stress is given by

$$\underset{\sim}{\sigma}(t) = -p\underset{\sim}{I} - \int_{-\infty}^t \mu_1(t - \tau) \underset{\sim}{J}_t(\tau) d\tau \quad . \tag{14}$$

It will emerge in section 3 that (14) is incapable of providing a satisfactory explanation of the non-monotonic response of ice

in a constant strain-rate test because of its linear dependence of stress on relative strain history. This difficulty is avoided by the more general version

$$\underset{\sim}{\sigma}(t) = -p\underset{\sim}{I} - \frac{1}{3}\int_{-\infty}^{t} K_{,2}[\frac{1}{3}(I_{\underset{\sim}{J}} - 3), t - \tau]\underset{\sim}{J}_t(\tau)d\tau , \quad (15)$$

where the kernel K is defined by

$$K[\frac{1}{3}(I_{\underset{\sim}{J}} - 3) , t] = 3\mu_1(I_{\underset{\sim}{J}}, t) \quad (16)$$

and $K_{,1}$ and $K_{,2}$ denote the partial derivatives of the kernel K with respect to its first and second arguments, respectively.

3. <u>Small Strain Uniaxial Tests</u>

In this section a one-dimensional constitutive equation is derived from equation (15) which can be applied to the uniaxial experimental tests discussed previously. In a uni-axial stress test

$$\underset{\sim}{\sigma}(t) = \begin{pmatrix} \sigma(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \quad (17)$$

The corresponding deformation gradient history is

$$F(t) = \begin{cases} 1 + \varepsilon(t) & 0 & 0 \\ 0 & 1 - \tfrac{1}{2}\varepsilon(t) & 0 \\ 0 & 0 & 1 - \tfrac{1}{2}\varepsilon(t) \end{cases}, \qquad (18)$$

which approximately satisfies the incompressibility constraint $(4)_2$ provided that the small strain condition

$$\varepsilon(t) \ll 1 \qquad (19)$$

holds for the longitudinal strain $\varepsilon(t)$. The substitution of (18) into the constitutive equation (15) via the relation (2) leads to the two scalar equations

$$\sigma_{11} = -p - \frac{1}{3} \int_{-\infty}^{t} K_{,2}[\varepsilon_t(\tau)^2, \, t - \tau][1 - 2\varepsilon_t(\tau)]d\tau \qquad (20)$$

and

$$\sigma_{22} = \sigma_{33} = -p - \frac{1}{3}\int_{-\infty}^{t} K_{,2}[\varepsilon_t(\tau)^2, \, t - \tau][1 + \varepsilon_t(\tau)]d\tau, \qquad (21)$$

where

$$\varepsilon_t(\tau) = \varepsilon(\tau) - \varepsilon(t) \qquad . \qquad (22)$$

The result

$$\sigma(t) = \int_{-\infty}^{t} K_{,2}[\varepsilon_t(\tau)^2, \, t - \tau]\varepsilon_t(\tau)d\tau \qquad (23)$$

now follows through the elimination of the pressure between (17), (20) and (21).

As an example of how the uniaxial constitutive equation (23) can be specialised to a specific test consider the constant strain test given by the linear strain history

$$\left.\begin{array}{ll} \varepsilon(t) = 0 \quad, & t < 0 \quad, \\ \\ \varepsilon(t) = \overline{\varepsilon} \quad, & t > 0 \quad, \end{array}\right\} \quad . \tag{24}$$

The substitution of (24) into (22) and (23) gives the expression

$$\sigma(t) = K(\varepsilon^{-2}, t)\overline{\varepsilon} \quad, \quad t > 0 \quad, \tag{25}$$

for the uniaxial stress, assuming the physically expected, monotonic, long-time behaviour

$$K(X, t) \to 0 \quad \text{and} \quad K_{,2}(X, t) \to 0 \quad \text{as} \quad t \to \infty \tag{26}$$

which must hold.

4. Constant Strain-Rate Test

The constant strain-rate test is specified by the linear strain history

$$\left.\begin{array}{ll} \varepsilon(t) = 0 \quad, & t < 0 \quad, \\ \\ \varepsilon(t) = rt \quad, & t > 0 \quad, \end{array}\right\} \tag{27}$$

where r is the strain rate. Using the alternative form

$$\sigma(t) = \int_{-\infty}^{t} \{K[\varepsilon_t(\tau)^2, t-\tau] + 2\varepsilon_t(\tau)^2 K_{,1}[\varepsilon_t(\tau)^2, t-\tau]\}\varepsilon_t'(\tau)d\tau \quad (28)$$

of equation (23), obtained by integrating by parts, with the dash denoting partial differentiation with respect to τ, the history (27) gives the relation

$$\sigma(t) = \int_{0}^{t} \{K[r^2(t-\tau)^2, t-\tau] + 2r^2(t-\tau)^2 K_{,1}[r^2(t-\tau^2), t-\tau]\}r d\tau \quad (29)$$

The results of the constant strain-rate test are usually recorded as a one parameter family of curves of stress measured against strain (rt), the parameter being the strain rate. Thus the experimental data is recorded as a function of two variables which can be expressed in terms of the model by substituting $u = r(t-\tau)$ in (29) to obtain

$$\tilde{\sigma}(\varepsilon, r) = \int_{0}^{\varepsilon} [K(u^2, r^{-1}u) + 2u^2 K_{,1}(u^2, r^{-1}u)] \, du \quad (30)$$

The relation (30) can be inverted to give

$$K(\varepsilon^2, t) = \varepsilon^{-1} \int_{0}^{\varepsilon} \tilde{\sigma}_{,1}(u, t^{-1}u) du \quad (31)$$

where $\tilde{\sigma}_{,1}$ and $\tilde{\sigma}_{,2}$ are, respectively, the partial derivatives of $\tilde{\sigma}$ with respect to its first and second arguments.

Thus the kernel can, in principle, be determined from the constant strain-rate test by means of equation (31).

When the results given above are specialised to the simple model (14), the kernel is a function of t only and equation (31) can be replaced by

$$K(t) = \tilde{\sigma}_{,1}(rt, r) .$$ (32)

It can be shown that this model is equivalent to the linear viscoelastic relation employed by Spring and Morland (1983) to test the accuracy of their numerical algorithm and, as can be seen from Spring and Morland (1983, Fig. 3), the model cannot adequately represent the behaviour of ice in this test because the strain at maximum stress is proportional to the constant strain-rate r and the maximum stress is also proportional to r, which contradicts the experimental evidence (see, for example, Mellor and Cole, 1983) that the strain at maximum stress (typically around 1%) is largely independent of r and the maximum stress is a non-linear function of r , tending to a finite value as r → ∞ . There is a further, more important, reason for the failure of the "neo-Hookean" fluid model (14). When equation (32) is used to determine the kernel from a typical constant strain rate response (e.g. Spring and Morland, 1983, Fig. 3(c)), K(t) is found to be non-monotonic and decreases through zero attaining a negative minimum value before decaying to zero. This kernel is physically unacceptable because it predicts absurd reponse in the relaxation test equation (25) and it is not a monotonic decreasing function

of time (a reasonable requirement for a one-dimensional
relaxation function, although it is a matter of some controversy
whether non-monotonic relaxation functions are compatible
with the principles of thermodynamics).

Equation (25) shows how the present model overcomes the
difficulties encountered above. Representing the constant
strain test data (if obtainable) by a function $\sigma^*(\varepsilon, t)$,
where

$$\sigma^*(\varepsilon, t) = K(\varepsilon^2, t)\varepsilon \tag{33}$$

is given by equation (25), the connection

$$\sigma^*_{,1}(\varepsilon, t) = \tilde{\sigma}_{,1}(\varepsilon, t^{-1}\varepsilon) \tag{34}$$

now follows from (31) and (33). It follows from (34) that
if $\tilde{\sigma}(\varepsilon, t)$ takes its maximum value at a strain of 1% then
so does $\sigma^*(\varepsilon, t)$, i.e. the non-monotonicity of the constant
strain rate response is modelled through the dependence of the
kernel on strain, the value of 1% representing a failure
strain in both the constant strain-rate and constant strain
tests. This is in direct contrast to the linear fluid model
(14), where the mathematical structure forces the modelling of
the non-monotonicity of the constant strain-rate response to
be done through the time dependence of the kernel.

Examination of equation (31) shows that the kernel K can
only be determined by the constant strain-rate test in a

region of the (ϵ, t)-plane defined by

$$r_{max}\, t \geq |\epsilon| \geq r_{min}\, t \tag{35}$$

and

$$|\epsilon| \leq \epsilon_{max} , \tag{36}$$

where ϵ_{max} is the maximum attainable experimental strain and r_{max} and r_{min} represent the limits within which the strain rate falls.

It is not possible to evaluate the single integral relation of Spring and Morland (1983, eq. 20, p.189) in the case of the constant strain test because this model will not admit a strain jump, but it is possible to examine the response for the strain history

$$r(t) = \begin{cases} 0 , & t < 0 , \\ rt , & 0 \leq t \leq t_1 , \\ rt_1, & t_1 < t , \end{cases} \tag{37}$$

where the time t_1 is small and r is constant. The substitution of (37) into Spring and Morland's equation gives

$$\sigma(t) = T(r, t) - T(r, t-t_1) \tag{38}$$

for $t > t_1$, where $T(r, t)$ is the stress at time t in a uniaxial compression test at constant strain-rate r.

A sketch of a typical stress relaxation curve given by

equation (38) is presented in figure 2. It can be seen from the graph of $\sigma(t)$ that the response is not only non-monotonic but the stress actually changes sign before decaying to a constant value of zero. This response is similar to the constant strain response exhibited by the "neo-Hookean" fluid model (14) when the modelling of non-monotonic constant strain rate behaviour is attempted. Spring and Morland's model does, however, give a good account of the variation of the maximum strain-rate but due to the kernel's dependence on strain rate rather than strain precludes the modelling of non-monotonic constant strain-rate response other than through the time dependence of the kernel. It is for this reason that the model of Spring and Morland is rejected in favour of the present approach.

5. Constant Stress Test. I. Approximate Solution

The problem of the constant stress stess may be summarised in the following question: given the uniaxial stress history

$$\sigma(t) = \begin{cases} 0 , & t < 0 , \\ \overline{\sigma} , & t \geq 0 , \end{cases} \tag{39}$$

where $\overline{\sigma}$ is the constant stress, what form does the strain $\varepsilon(t)$ $(t > 0)$ take? An approximate solution is obtained by first addressing the more general question of finding the strain $\varepsilon(t)$ $(t > 0)$ generated by a stress field which vanishes for $t < 0$ but is arbitrary for $t > 0$; that is,

$$\sigma(t) = 0 , \quad t < 0 . \tag{40}$$

Making the substitution

$$u = - \varepsilon_t(\tau) \tag{41}$$

in equation (28), noting that (40) implies that $\varepsilon(t) = 0$ for $t < 0$, gives for $t > 0$,

$$\sigma(t) = \int_0^\varepsilon \{K[u^2, t(\varepsilon) - t(\varepsilon-u)] + 2u^2 K_{,1}[u^2, t(\varepsilon) - t(\varepsilon-u)]\} du . \tag{42}$$

It now follows from (30) that

$$K(\varepsilon^2, t) + 2\varepsilon^2 K_{,1}(\varepsilon^2, t) = \tilde{\sigma}_{,1}(\varepsilon, t^{-1}\varepsilon) , \tag{43}$$

and so (42) may be expressed in the simpler form

$$\sigma(t) = \int_0^\varepsilon \tilde{\sigma}_{,1}[u, \frac{u}{t(\varepsilon) - t(\varepsilon-u)}] du . \tag{44}$$

Now make the assumption that <u>events in the recent past dominate the response of the material to a greater extent than events in the distant past</u>. Then

$$\frac{t(\varepsilon) - t(\varepsilon-u)}{u} \simeq \frac{1}{\dot{\varepsilon}[t(\varepsilon)]} \tag{45}$$

should represent a reasonable approximation in the integral (44), and leads to the approximate relation

$$\sigma(t) = \tilde{\sigma}(\varepsilon, \dot{\varepsilon}) . \tag{46}$$

Equation (46) is a first order ordinary differential equation for the strain $\varepsilon(t)$. It may also, however, be regarded as an approximate "equation of state", replacing the integral relation (23) when suitable circumstances apply.

It is interesting to note that, when the stress history (39) is employed in (46), the constant stress data is given by the one-parameter family of curves in the $(\varepsilon, \dot{\varepsilon})$ plane represented by

$$\tilde{\sigma}(\varepsilon, \dot{\varepsilon}) = \overline{\sigma} \quad , \tag{47}$$

$\overline{\sigma}$ being the parameter. Furthermore, since the constant strain rate data is represented by the family

$$\sigma = \tilde{\sigma}(\varepsilon, r) \tag{48}$$

of curves in the (σ, ε) plane, it can be verified that the conjectures of Mellor and Cole (1982, Table 1, p.206), connecting the constant strain rate and constant stress data, are borne out by (47) and (48). Furthermore, the relationship between the constant stress and constant strain rate data embodied in (47) and (48) has recently been verified experimentally by Mellor and Cole (1983) to a reasonable degree of accuracy. We are thus led to the conclusion that the constant strain rate and constant stress data represent essentially the same information.

Equation (47) is, however, only an approximate solution to

the problem of the constant stress test and a better correlation between the constant stress data and the constant strain-rate data should be obtained from a numerical solution of the integral equation associated with the constant stress test.

6. The Structure of the Kernel

In this section the mathematical structure of the kernel $K(\varepsilon^2, t)$ is examined with the aid of the assumption that the constant strain-rate data can be expressed in a separable form

$$\tilde{\sigma}(\varepsilon, r) = S(\varepsilon)\sigma_{max}(\tau) . \tag{49}$$

This assumption is consistent with the experimental observation that the stress-strain curves for different constant strain-rates are non-intersecting. The function $S(\varepsilon)$ is referred to as the shape function and is sketched in figure 3. $\sigma_{max}(r)$ is the maximum stress, which is sketched in figure 4. It should be noted that no attempt has been made to include the experimentally observed fine structure to the left of the peak in the shape function because this is believed to be due to internal cracking (Mellor and Cole, 1982), which is outside the scope of this present model.

When (49) is substituted into (34) and the result is integrated by parts the relation

$$\sigma^*(\varepsilon, t) = S(\varepsilon)\sigma_{max}(t^{-1}\varepsilon) - \int_o^{\varepsilon} S(u)\sigma_{max}{'}(t^{-1}u)t^{-1}du \tag{50}$$

is obtained. When the substitution $tv = u$ is made in the integral on the right-hand side of (50) it is apparent that

$$\sigma^*(\epsilon, 0) = \sigma_\infty S(\epsilon) \quad . \tag{51}$$

By definition, $\sigma^*(\epsilon, 0)$ is the instantaneous elastic response and, apart from the constant multiplier $\sigma_\infty = \lim_{r \to \infty} \sigma_{max}(r)$, it is the same as the shape function so that reference may again be made to figure 3. It is now possible to sketch the initial value $K(\epsilon^2, 0)$ of the kernel and this is given in figure 5.

It may also be inferred from the result

$$\sigma^*,_1(\epsilon, t) = S'(\epsilon)\sigma_{max}(t^{-1}\epsilon) \quad , \tag{52}$$

which is a consequence of (50), that $\sigma^*,_1(\epsilon, t)$ $(t > 0)$ is of a similar nature, viewed as a function of ϵ , to $\sigma^*,_1(\epsilon, 0)$, with the exception that the former vanishes at $\epsilon = 0$. Therefore it follows that $K(\epsilon^2, t)$, which is given by (33), has a similar shape to the initial kernel $K(\epsilon^2, 0)$ shown in figure 5.

Acknowledgement

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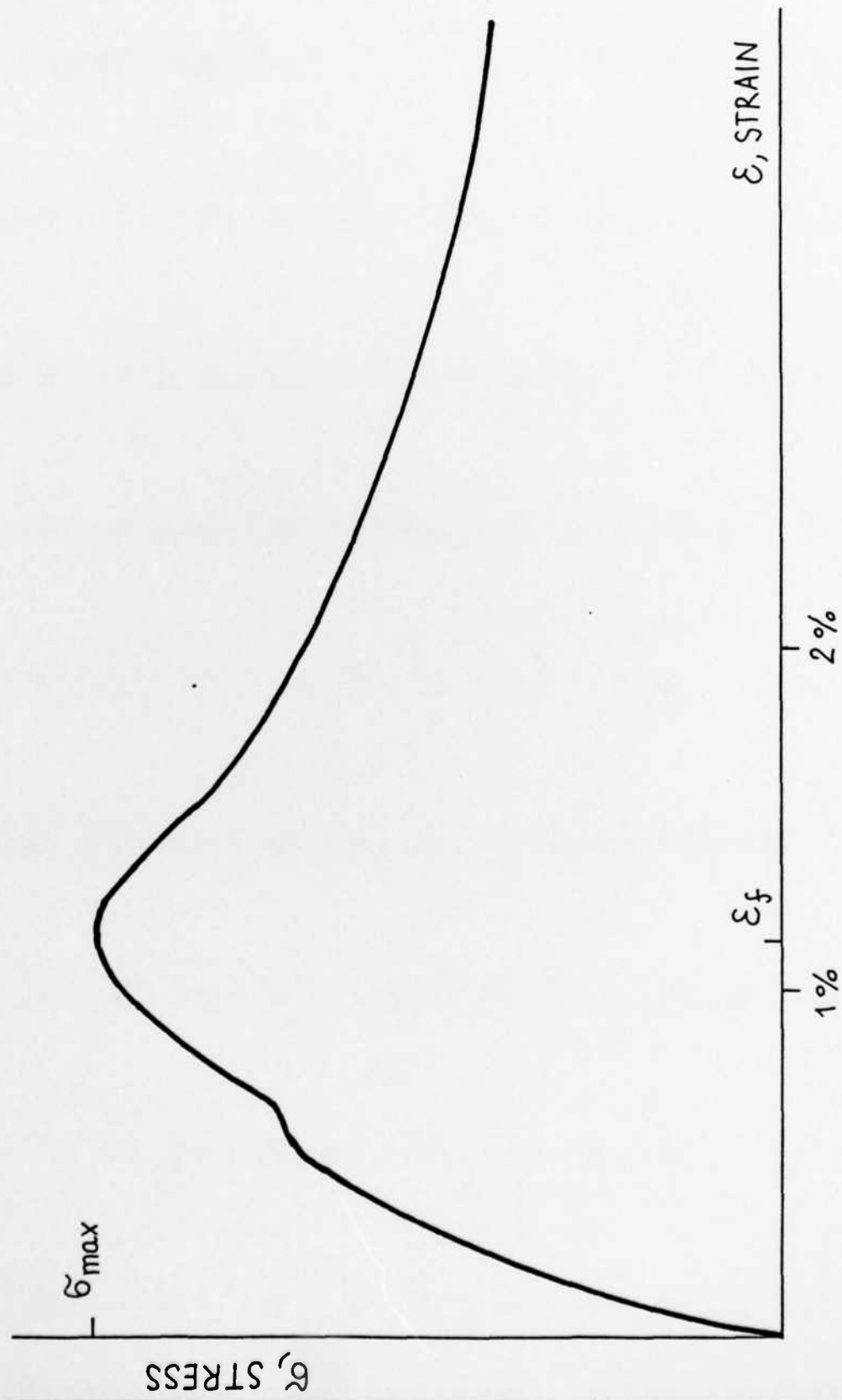
FIGURE 1 COMPLETE STRESS-STRAIN CURVE FOR CONSTANT STRAIN RATE (CD) TEST
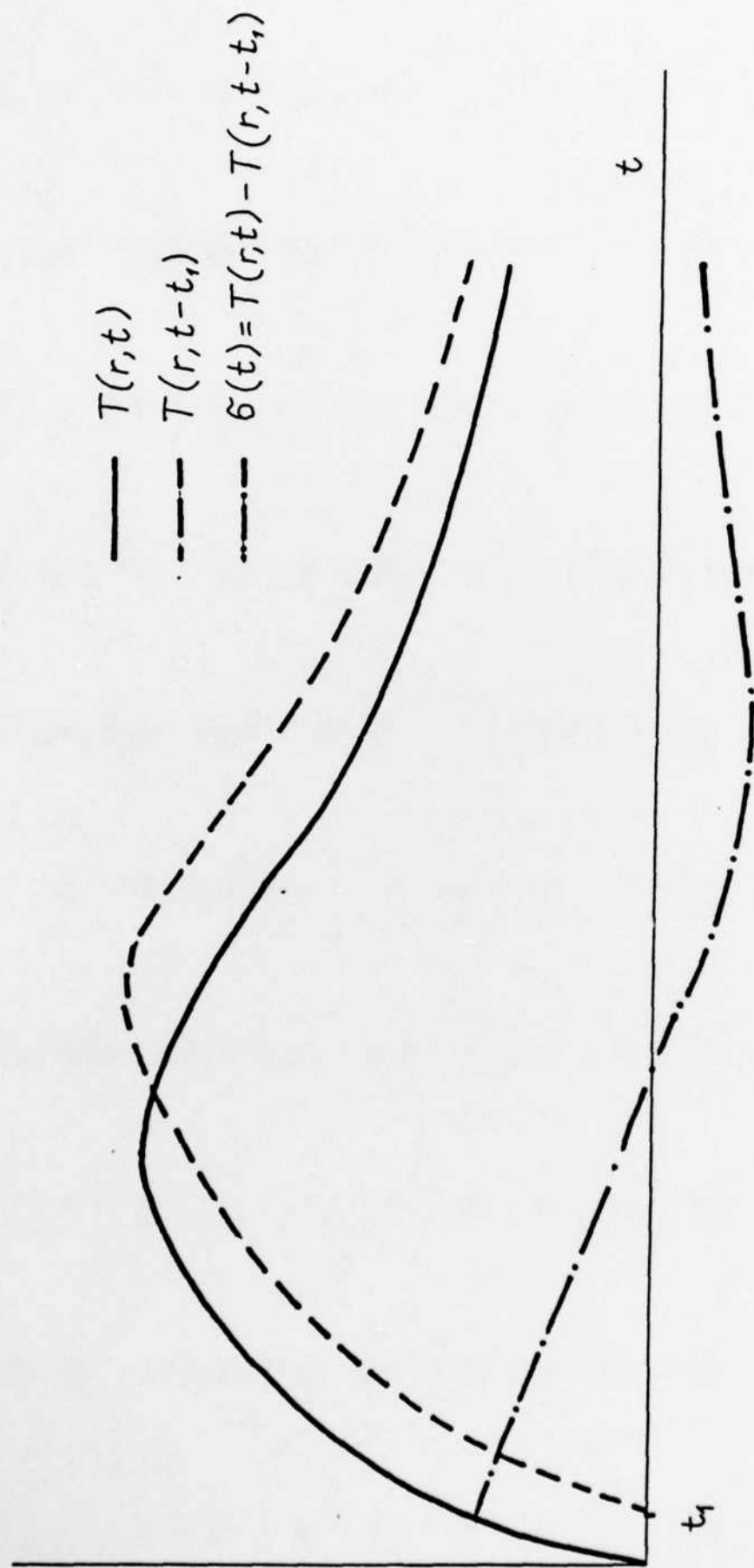
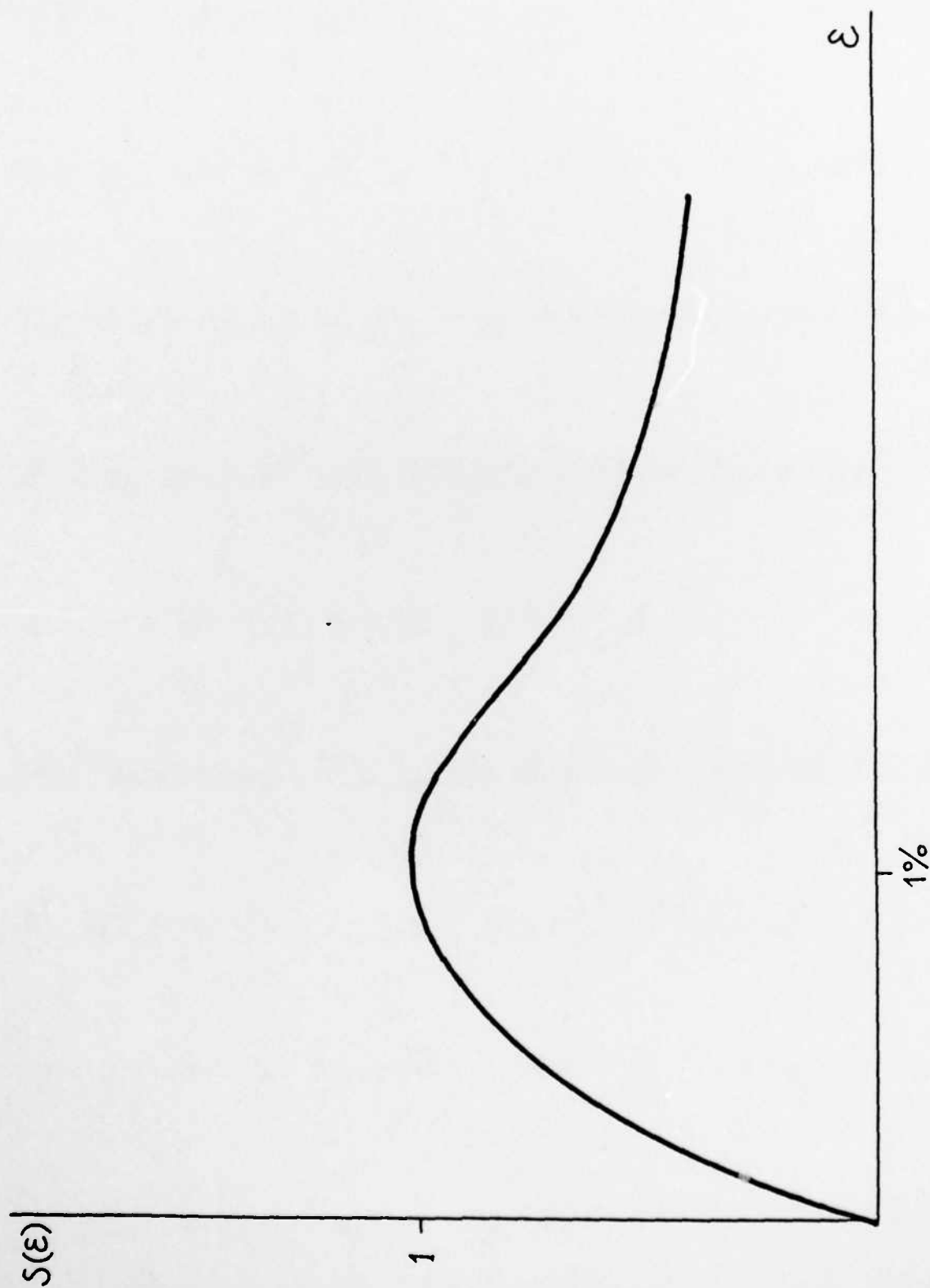FIGURE 2   STRESS RELAXATION RESPONSE FOR SPRING AND MORLAND INTEGRAL
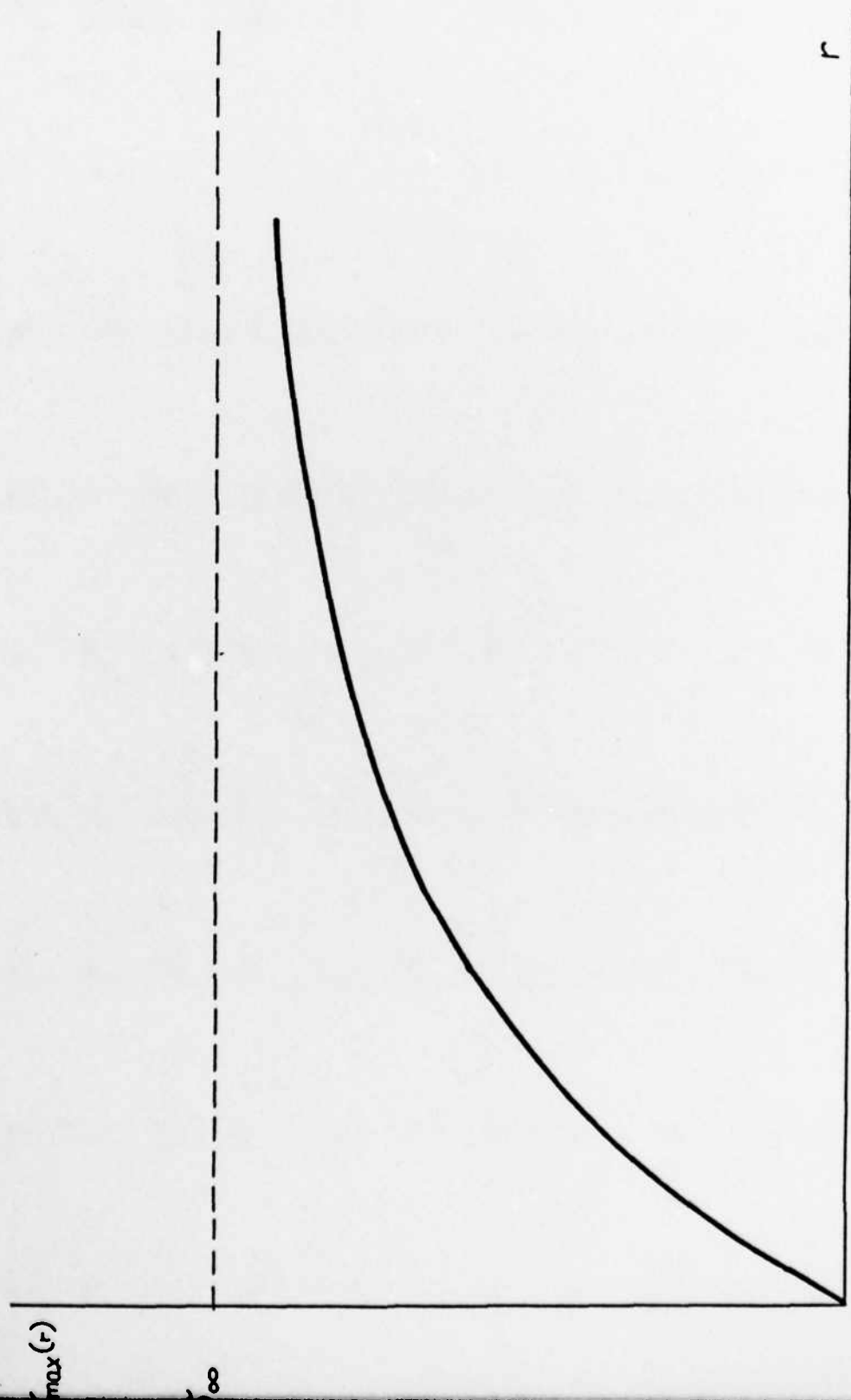REPRESENTATION

FIGURE 3   THE SHAPE FUNCTION   S(ε)

FIGURE 4   THE MAXIMUM STRESS $\sigma_{max}(r)$
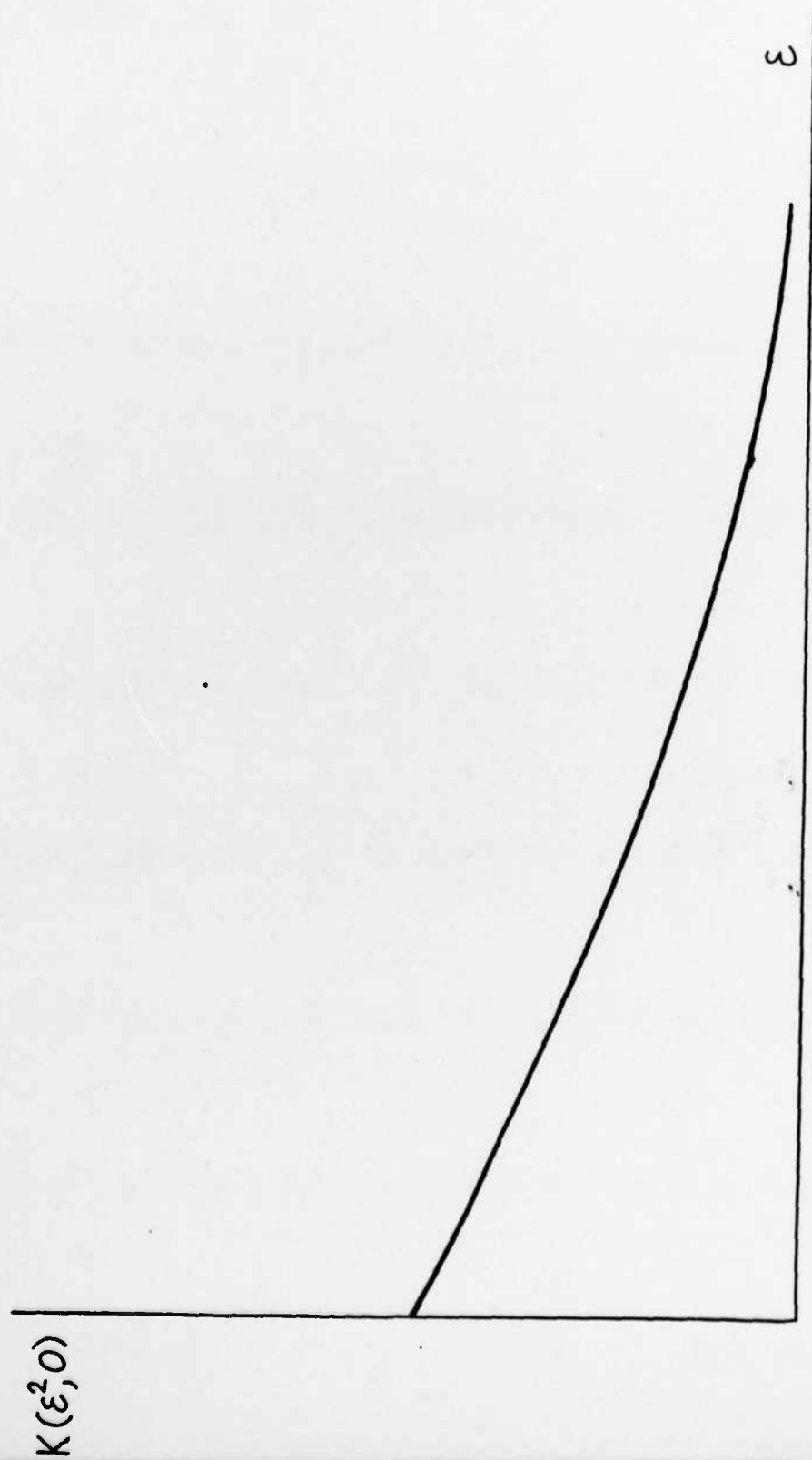
$K(\varepsilon^2,0)$

ε

FIGURE 5   THE INITIAL KERNEL $K(\varepsilon^2,0)$

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